

# MEAN-FIELD GAMES WITH CONTROLLED JUMPS

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**ABSTRACT.** We study a family of mean-field games (MFGs) with a controlled jump component. We establish existence of a solution in a relaxed version of the MFG and give conditions under which the optimal strategies are in fact Markovian. The proofs rely upon the notions of relaxed controls and martingale problems. Finally, we apply the general theory to a simple illiquid interbank market model and provide some numerical results.

*Keywords:* mean-field games, jump measures, controlled martingale problem, relaxed controls, illiquid interbank model.

## 1. INTRODUCTION

Mean field games (MFGs, henceforth) were introduced by Lasry and Lions in [LL06a, LL06b, LL07] and, independently, by Huang and co-authors in [HMC06], as optimization problems approximating large population symmetric stochastic differential games, where the interaction between the players is of mean field type. A solution of the limit MFG allows to construct approximate Nash equilibria for the corresponding  $N$ -player games if  $N$  is large enough; see, e.g., [HMC06], [KLY11], [CD13a], [CD13b], [CL15] as well as the recent book [Car16]. This approximation result is also practically relevant since a direct computation of Nash equilibria in the  $N$ -player game with  $N$  very large is usually not feasible even numerically, due to the curse of dimensionality. Moreover, MFGs represent a very flexible framework for applications in various areas including but not limited to finance, economics and crowd dynamics (see [GLL11, Car16] for a good sample of applications), which partly explain the increasing literature on the subject.

In this paper, we study a family of MFGs with controlled jumps, that can be shortly described as follows. Let  $T > 0$  be a finite time horizon and let  $(m(t))_{t \in [0, T]}$  a deterministic right-continuous with left limit (henceforth càdlàg) function. We consider the controlled state variable  $X = X^\gamma$  following the dynamics

$$(1) \quad dX_t = b(t, X_t, m(t))dt + \sigma(t, X_t)dW_t + \lambda(m(t-))\gamma_t d\tilde{N}_t, \quad t \in [0, T],$$

where  $\gamma_t$  represents a control process, taking values in a fixed action space  $A$ ,  $W$  is a standard Brownian motion and  $\tilde{N}$  is some (compensated) jump process. Moreover, we assume that  $W$  and  $N$  are independent.

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Given some running cost  $f$  and some final cost  $g$ , the aim is to find a control  $\hat{\gamma}$  solving the following minimization problem

$$(2) \quad \inf_{\gamma} \mathbb{E} \left[ \int_0^T f(t, X_t, m(t), \gamma_t) dN_t + g(X_T, m(T)) \right]$$

over all control processes  $\gamma$  as above and such that the so-called mean-field condition is fulfilled: the function  $m$  has to be equal to the expected value of the optimally controlled dynamic  $\hat{X} = X^{\hat{\gamma}}$  at each time, that is  $m(t) = \mathbb{E}[\hat{X}_t]$  for all  $t \in [0, T]$ .

This model is the natural limit of a symmetric nonzero-sum  $N$ -player game, where each player controls the jumps of her/his own private state and the states are coupled through their empirical mean. In more detail, consider the following dynamics for the private state,  $X^i$ , of player  $i = 1, \dots, N$ :

$$(3) \quad dX_t^i = b(t, X_t^i, \bar{X}_t)dt + \sigma(t, X_t^i)dW_t^i + \lambda(\bar{X}_{t-})\gamma_t^i d\tilde{N}_t^i, \quad t \in [0, T],$$

where  $\gamma^i$  is the control of player  $i$ ,  $\bar{X}_t = (1/N) \sum_{j=1}^N X_t^j$  is the empirical mean of the vector  $(X_t^1, \dots, X_t^N)$  of the private states of all players,  $(W^1, \dots, W^N)$  is an  $N$ -dimensional Brownian motion and  $(N^1, \dots, N^N)$  is an  $N$ -dimensional (compensated) jump process. The goal of each player  $i$  is minimizing some objective function, given by

$$(4) \quad \mathbb{E} \left[ \int_0^T f(t, X_t^i, \bar{X}_t, \gamma_t^i) dN_t^i + g(X_T^i, \bar{X}_T) \right] = \mathbb{E} \left[ \int_0^T f(t, X_t^i, \bar{X}_t, \gamma_t^i) \lambda(\bar{X}_t) dt + g(X_T^i, \bar{X}_T) \right],$$

over her/his controls. This is a symmetric stochastic differential game, where the agents interact through the empirical mean of their private states, entering in both the drift and the jump component. According to mean-field game theory, we expect that as the number of player gets larger and larger, the  $N$ -player game just described tends in some sense to the minimization problem in (1), that is the solution to the latter would provide a good approximation of some Nash equilibrium in the  $N$ -player game. In the present paper, we focus on the existence of solutions for the limit MFG and we postpone to future research the other equally important issues of uniqueness of MFG solution and approximation of Nash equilibria of the  $N$ -player game when  $N$  is large.

While the uncontrolled counter-part of MFG, that is particle systems and propagation of chaos for jump processes, has been thoroughly studied in the probabilistic literature (see, e.g., [Gra92, JMW08] and the very recent preprint [ADPF]), MFGs with jumps have not attracted much attention so far. Indeed, most of the existing literature focuses on non-linear dynamics with continuous paths, with the exception of few papers such as [HAA14] and [KLY11]. The former deals with stochastic control of McKean-Vlasov type (see [CDL13] for a comparison between MFG and McKean-Vlasov control), whereas the latter use methods based on potential theory and nonlinear Markov processes. The approach we use in this paper is based on weak formulation of stochastic controls, relaxed controls and martingale problems, which is very much inspired by Lacker [Lac15]. More in detail, our main contribution can be summarized as follows:

- We extend the relaxed approach to MFGs introduced by Lacker [Lac15] to a simple class of MFGs with controlled jump component described above, by establishing an existence result under some boundedness assumptions on the coefficients of the state variable and on the cost functional. The issue of uniqueness and approximation of the  $N$ -player games by means of the MFG solution is postponed to future research.

- We illustrate the relevance of this class of MFGs by means of an illiquid interbank market model, inspired by the systemic risk model proposed by Carmona and co-authors in [CFS15]. Within this model, we can compute explicitly the Nash equilibrium and see the convergence to the solution of the limit MFG. Moreover, we perform some numerical experiments showing the role of illiquidity in driving the evolution over time of the controls and the state variables.

The paper is structured as follows: in Section 2 we provide the relaxed version of the MFG with controlled jumps together with all the assumptions on the coefficients, while in Section 3 we state and prove the main existence results. In Section 4 we study Nash equilibria and their convergence towards the MFG solution in the illiquid interbank model; some numerical experiments are also performed. Finally, the paper ends with an appendix, collecting most of the technical results used in the proofs of the main theorems.

## 2. THE RELAXED MFG PROBLEM WITH CONTROLLED JUMPS

Following the approach in [Lac15], we are going to study a relaxed version of the MFG briefly described in the introduction. Slightly more precisely, we will re-define the state variable and the controls on a suitable canonical space supporting all the randomness sources involved in the SDE with jumps above, so that the solution to the MFG will be identified with a probability measure  $P$  on that space, that can be seen as the joint law of the pair state/control as in [Lac15]. Therefore, finding a relaxed solution to the MFG above will boil down to finding a fixed point for a suitably defined set-valued map. The rest of this section sets up the main assumptions on the state variable and the cost functions as well as the precise definition of relaxed mean-field game with controlled jumps, while the issue of existence will be addressed in the next section. For the sake of simplicity, we consider only the one-dimensional case, i.e. we assume that the state variable  $X$  takes real values.

**2.1. Assumptions.** We start with some notation that will be used throughout the whole paper. Let  $D = D([0, T]; \mathbb{R})$  denote the set of all càdlàg functions  $m: [0, T] \rightarrow \mathbb{R}$ , endowed with the Skorohod topology. Given some metric space  $(S, d)$ ,  $\mathcal{P}(S)$  denotes the set of all probability measures defined on the measurable space  $(S, \mathcal{B}(S))$ , where  $\mathcal{B}(S)$  is the Borel  $\sigma$ -field of  $S$ . Moreover,  $\mathcal{P}^p(S)$ , for some  $p \geq 1$ , denotes the set all  $P \in \mathcal{P}(S)$  such that  $\int_S d(x, x_0)^p P(dx) < \infty$  for some (hence for all)  $x_0 \in S$ . Product spaces will always be endowed with the product  $\sigma$ -fields.

In what follows we will make use of the following standing assumptions on the coefficients  $b, \sigma, \lambda$ , the costs  $f, g$  and the initial distribution,  $\chi \in \mathcal{P}(\mathbb{R})$ , of the state process  $X$ .

- Assumption A.** (A.1)  $\lambda: \mathbb{R} \rightarrow \mathbb{R}_+$  is a continuous function, bounded by some constant  $c_\lambda$ .  
 (A.2) The functions  $b: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\sigma: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $f: [0, T] \times \mathbb{R} \times \mathbb{R} \times A \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are jointly continuous in all their variables.  
 (A.3) There exists a constant  $c_1 > 0$  such that for all  $(t, m) \in [0, T] \times \mathbb{R}$  and for all  $x, y \in \mathbb{R}$ , we have

$$\begin{aligned} |b(t, x, m) - b(t, y, m)| + |\sigma(t, x) - \sigma(t, y)| &\leq c_1 |x - y|, \\ |b(t, x, m)| + |\sigma(t, x)| &\leq c_1. \end{aligned}$$

Furthermore,  $\sigma$  is bounded away from zero, i.e. there exists  $c_\sigma > 0$  such that

$$\sigma(t, x) \geq c_\sigma \quad \text{for each } (t, x) \in [0, T] \times \mathbb{R}.$$

(A.4) There exists some positive constant  $c_2 > 0$  such that for each  $(t, x, m, \alpha) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times A$

$$|g(x, m)| + |f(t, x, m, \alpha)| \leq c_2.$$

Without loss of generality we can assume  $c_1 = c_2$ .

(A.5) The control space  $A$  is a compact subset of  $\mathbb{R} \setminus \{0\}$ . Let  $\alpha_M = (\max_A \alpha) \vee 1$ .

Few comments on the assumptions above are in order. Conditions (A.2) and (A.3) ensure the existence of a unique strong solution of the SDE (7) governing the evolution of the state variable. Assumptions (A.1), (A.2) and (A.4) are widely used in Lemma A.4 and Lemma A.5, which establish good compactness and continuity properties needed in the fixed point argument. Lastly, assumption (A.5) gives in particular that the space of controls is a complete separable metric space, which is needed to apply tightness criteria as in Proposition A.2.

*Remark 1.* By adapting to our setting the arguments used in [Lac15, Section 5], our results could in principle be extended to the case when  $b$  and  $\sigma$  are linear functions on their whole domain. The main idea is to work with an approximation of these functions, namely their truncated and therefore bounded versions. Then, by a convergence argument, it must be shown that the limit of the mean field game solutions found in the truncated setting is indeed a solution for the unbounded case.

**2.2. Relaxed controls and admissible laws.** A relaxed control is a measure  $\Gamma$  on the set  $[0, T] \times A$ , equipped with the product  $\sigma$ -field  $\mathcal{B}([0, T] \times A)$ , such that its first marginal equals the Lebesgue measure, i.e.  $\Gamma([s, t] \times A) = t - s$ , for all  $0 \leq s \leq t \leq T$ , and

$$\int_{[0, T] \times A} |\alpha|^2 \Gamma(dt, d\alpha) < \infty.$$

The set of all relaxed controls will be denoted by  $\mathcal{V} = \mathcal{V}[A]$ . We endow  $\mathcal{V}$  with the (modified) 2-Wasserstein metric  $d_{\mathcal{V}}$ , given by

$$d_{\mathcal{V}}(\Gamma, \Gamma') = d_W\left(\frac{\Gamma}{T}, \frac{\Gamma'}{T}\right)$$

where for any measure  $\mu, \nu \in \mathcal{P}^2(A)$

$$\begin{aligned} d_W(\mu, \nu) &= \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{A \times A} |x - y|^2 \pi(dx, dy) \right\}, \\ \Pi(\mu, \nu) &= \{ \pi \in \mathcal{P}(A \times A) : \pi \text{ has marginals } \mu, \nu \}. \end{aligned}$$

$\mathcal{V}$  is a complete separable metric space and, since the action space  $A$  is compact (ref. Assumption (A.5)), so is  $\mathcal{V}$ .

Every relaxed control  $\Gamma \in \mathcal{V}$  can be related to a unique (up to a.e. equality) measure-valued map  $t \mapsto \Gamma_t \in \mathcal{P}(A)$  such that  $\Gamma(dt, d\alpha) = dt \Gamma_t(d\alpha)$ . Moreover a control  $\Gamma \in \mathcal{V}$  is said to be strict if  $\Gamma_t = \delta_{\gamma(t)}$  for some  $A$ -valued measurable stochastic process  $\gamma_t$  for  $t \in [0, T]$ , where  $\delta_x$  denotes the Dirac delta function at the point  $x$ .

Let  $\Omega[A] = \mathcal{V} \times D$  be endowed with its Borel  $\sigma$ -field. An element of  $\Omega[A]$  is denoted by  $(\Gamma, X)$ , and, with a slight abuse of notation, we denote  $\Gamma$  (resp.  $X$ ) its projection onto  $\mathcal{V}$  (resp.  $D$ ).

Let  $L$  be the linear integro-differential operator defined on  $C_0^\infty(\mathbb{R})$ , i.e. the set of all infinitely differentiable functions  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  having compact support, by

$$\begin{aligned} (5) \quad L\phi(t, x, m, \Gamma) &= b(t, x, m(t))\phi'(x) + \frac{1}{2}\sigma^2(t, x)\phi''(x) \\ &\quad + \lambda(m(t-)) \int_A [\phi(x + \alpha) - \phi(x) - \phi'(x)\alpha] \Gamma_t(d\alpha) \end{aligned}$$

for each  $(t, x, m, \Gamma) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{V}$ . For each  $\phi \in C_0^\infty(\mathbb{R})$  and for each function  $m \in D$ ,  $\mathcal{M}_t^{m, \phi}: \Omega[A] \rightarrow \mathbb{R}$  is the operator defined by

$$(6) \quad \mathcal{M}_t^{m, \phi}(\Gamma, X) = \phi(X_t) - \int_{[0, t] \times A} L\phi(s, X_s, m(s), m(s), \Gamma_s) ds, \quad t \in [0, T].$$

**Definition 1.** Let  $m$  be a given function in  $D$  and  $P$  be a probability measure in  $\mathcal{P}^2(\Omega[A])$ . We say that  $P$  is an admissible law if it satisfies the following conditions:

- (1)  $X_0 \sim \chi \in \mathcal{P}^2(\mathbb{R})$ ;
- (2)  $\mathcal{M}^{m, \phi} = (\mathcal{M}_t^{m, \phi})_{t \in [0, T]}$  is a  $P$ -martingale for each  $\phi \in C_0^\infty$ .

We denote by  $\mathcal{R}^2(m)$  the set of all admissible laws.

From now on, we will simplify the notation letting  $\mathcal{P}(\Omega[A]) = \mathcal{P}^2(\Omega[A])$  and  $\mathcal{R}(m) = \mathcal{R}^2(m)$ .

*Remark 2.* According to Definition 1,  $\mathcal{R}$  represents a set-valued correspondence  $\mathcal{R}: D \rightrightarrows \mathcal{P}(\Omega[A])$ . It is clear that for each càdlàg function  $m \in D$ ,  $\mathcal{R}(m)$  is nonempty if the martingale problem (6) admits at least one solution. The latter is guaranteed by the regularity Assumption A thanks to [MP92, Theorem 5]. Moreover,  $\mathcal{R}(m)$  is a convex set for each  $m \in D$ , i.e. any convex combination  $Q$  of probability measures in  $\mathcal{R}(m)$ , that is  $Q = \gamma P_1 + (1 - \gamma)P_2$  with  $\gamma \in [0, 1]$  and  $P_1, P_2 \in \mathcal{R}(m)$ , is still an element of  $\mathcal{R}(m)$ .

An application of [Kur11, Theorem 2.3] gives the following equivalent characterization of the elements of  $\mathcal{R}(m)$  as solutions of the martingale problem associated to the operator  $L$ . Notice that, even though in our case the dynamics of  $X$  is non-homogenous in time, Kurtz's result can be applied to the state variable  $X'_t = (t, X_t)$  with initial law  $\chi' = \delta_0 \otimes \chi$ .

**Lemma 2.1.** Given a function  $m \in D$ ,  $\mathcal{R}(m)$  is the set of all probability measures  $Q$  on  $\Omega[A]$  such that  $(\Omega[A], \mathcal{F}', \mathbb{F}' = \{\mathcal{F}'_t\}_{t \in [0, T]}, Q)$  is a filtered probability space satisfying the usual conditions and supporting an  $\mathbb{F}'$ -adapted process  $X$ , an  $\mathbb{F}'$ -adapted Brownian motion  $B$  and a Poisson random measure  $N$  on  $[0, T] \times A$ , with mean measure  $\mathcal{L} \times \Gamma_t$ , where  $\mathcal{L}$  denotes the Lebesgue measure, such that

$$Q \circ X_0^{-1} = \chi$$

and the state process  $X$  satisfies the following equation

$$(7) \quad X_t = X_0 + \int_0^t b(s, X_s, m(s)) ds + \int_0^t \sigma(s, X_s) dB_s + \int_{[0, t] \times A} \lambda(m(s-)) \alpha \tilde{N}(ds, d\alpha),$$

where, as usual,  $\tilde{N}$  denotes the compensated Poisson random measure, i.e.  $\tilde{N}(dt, d\alpha) = N(dt, d\alpha) - \Gamma_t(d\alpha)dt$ .

Note that under Assumption A, the SDE with jumps (7) admits a unique strong solution (see, e.g., [App09, Theorem 6.2.3]).

**2.3. Relaxed mean-field game.** For any càdlàg function  $m \in D$ , the objective cost function of the minimization problem is  $\mathcal{C}^m: \Omega[A] \rightarrow \mathbb{R}$  defined by

$$(8) \quad \mathcal{C}^m(\Gamma, X) = \int_{[0, T] \times A} f(t, X_t, m(t), \alpha) \Gamma(dt, d\alpha) + g(X_T, m(T)).$$

Thus, the relaxed MFG with controlled jump component consists in finding a probability measure  $P^* \in \mathcal{R}(m)$  so that the expected cost under  $P^*$  is minimal, i.e.

$$\int_{\Omega[A]} \mathcal{C}^m dP^* = \inf_{P \in \mathcal{R}(m)} \int_{\Omega[A]} \mathcal{C}^m dP.$$

We now focus on the problem of existence of such a  $P^*$ . Since this problem will be studied via a fixed point argument, we introduce the correspondence  $\mathcal{R}^*$  defined as

$$(9) \quad \begin{aligned} \mathcal{R}^*: D &\rightarrow \mathcal{P}(\Omega[A]) \\ m &\rightarrow \mathcal{R}^*(m) = \arg \min_{P \in \mathcal{R}(m)} J(m, P), \end{aligned}$$

where

$$(10) \quad \begin{aligned} J: D \times \mathcal{P}(\Omega[A]) &\rightarrow \mathbb{R} \cup \{\infty\} \\ (m, P) &\mapsto J(m, P) = \mathbb{E}^P[\mathcal{C}^m] = \int_{\Omega[A]} \mathcal{C}^m dP. \end{aligned}$$

**Definition 2.** A relaxed MFG solution is a probability distribution  $P \in \mathcal{P}(\Omega[A])$  such that  $P \in \mathcal{R}^*(\mathbb{E}^P[X])$ , i.e. it provides a fixed point for the set-valued map

$$D \ni m \mapsto \{\mathbb{E}^P[X] : P \in \mathcal{R}^*(m)\}.$$

A relaxed MFG solution is said to be Markovian (resp. strict Markovian) if the  $\mathcal{V}$ -marginal of  $P$ , i.e.  $\Gamma$ , satisfies  $P(\Gamma(dt, d\alpha) = dt\hat{\Gamma}(t, X_{t-})(d\alpha)) = 1$  for a measurable function  $\hat{\Gamma}: [0, T] \times \mathbb{R} \rightarrow \mathcal{P}(A)$  (resp.  $P(\Gamma(dt, d\alpha) = dt\delta_{\hat{\gamma}(t, X_{t-})}(d\alpha)) = 1$  for a measurable process  $\hat{\gamma}: [0, T] \times \mathbb{R} \rightarrow A$ ).

The following theorem summarizes our main result on the existence of a strict Markovian solution for the MFG with controlled jumps. It follows from a combination of Theorem 3.1 and Theorem 3.2 in the next section. We need to introduce one further assumption:

**Assumption C.** For all  $(t, x, m) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ , the set

$$(11) \quad K(t, x, m) := \{(b(t, x, m), \lambda(m)\alpha, f(t, x, m, \alpha)) : \alpha \in A, z \leq f(t, x, m, \alpha)\}$$

of  $\mathbb{R}^3$  is convex.

$L^1(\mathbb{R})$  stands for the set of all integrable real-valued functions defined on the real line, equipped with the Borel  $\sigma$ -field and the Lebesgue measure.

**Theorem 2.2.** Under Assumption A, there exists a relaxed MFG solution  $P$ . Moreover, if also Assumption C holds and if  $P$  satisfies

$$(12) \quad \sup_{t \in [0, T]} |\varphi_{X_t}| \in L^1(\mathbb{R}),$$

where  $\varphi_{X_t}(\zeta) := \mathbb{E}^P[e^{i\zeta \cdot X_t}]$ ,  $\zeta \in \mathbb{R}$ , is the characteristic function of  $X_t$  under  $P$ , there exists a strict Markovian MFG solution, i.e. the following two properties are satisfied:

- (1) there exists a càdlàg function  $m \in D$  and a measurable function  $\hat{\gamma}: [0, T] \times \mathbb{R} \rightarrow A$  which satisfy the following SDE

$$\begin{cases} dX_t = b(t, X_t, m(t))dt + \sigma(t, X_t)dW_t + \hat{\gamma}(t, X_{t-})d\tilde{N}_t \\ P \circ X_0^{-1} = \chi \end{cases}$$

where  $\chi$  is a fixed distribution in  $\mathcal{P}^2(\mathbb{R})$ , and satisfying

$$\mathbb{E}^P[X_t] = m_t \quad \text{for all } t \in [0, T].$$

(2) If  $(\Omega[A], \mathcal{F}', \mathbb{F}', P')$  is another filtered probability space,  $W'$  a one-dimensional  $\mathbb{F}'$ -Brownian motion,  $X'$  an  $\mathbb{F}'$ -adapted process,  $\gamma'$  an  $\mathbb{F}'$ -predictable process and  $N'$  a Poisson random measure with intensity measure  $(\lambda(m(t)))_{t \in [0, T]}$ , such that

$$\begin{cases} dX'_t = b(t, X'_t, m(t))dt + \sigma(t, X'_t)dW'_t + \gamma'_t d\tilde{N}'_t \\ P' \circ (X'_0)^{-1} = \chi \end{cases}$$

then

$$\begin{aligned} \mathbb{E}^P \left[ \int_0^T f(t, X_t, m(t), \hat{\gamma}(t, X_{t-}))dt + g(X_T, m(T)) \right] \\ \leq \mathbb{E}^{P'} \left[ \int_0^T f(t, X'_t, m(t), \gamma'_t)dt + g(X'_T, m(T)) \right]. \end{aligned}$$

*Remark 3.* The property (12) might not be very easy to verify in concrete examples and sufficient conditions on the coefficients of the state variable guaranteeing such an integrability conditions do not seem to be available in the literature. However, we observe that any non-Markovian MFG solution can be approximated by a sequence of strict Markovian controls, giving rise to a nearly optimal solution, using a version of the chattering lemma for stochastic control of jump-diffusions as in [Kus00, Theorems 3.3 and 3.4].

### 3. EXISTENCE OF A RELAXED MFG SOLUTION

In this section we prove the existence of a relaxed solution of our MFG with controlled jumps, and of a (strict) Markovian representation of it. The technical results that we use in the proof can be found in the Appendix A.

**Theorem 3.1.** *Under Assumption A, there exists a relaxed MFG solution as in Definition 2.*

*Proof.* According to Definition 2, a probability distribution  $P \in \mathcal{P}(\Omega[A])$  is a relaxed MFG solution if it provides a fixed point for the correspondence  $\mathcal{E}$  given by

$$(13) \quad \begin{aligned} \mathcal{E} : D &\rightarrow D \\ m &\mapsto \mathcal{E}(m) = \{ \mathbb{E}^P[X] : P \in \mathcal{R}^*(m) \} \end{aligned}$$

where  $\mathcal{R}^*$  denotes the correspondence defined in (9). In order to prove the existence of such a  $P$ , we apply the Kakutani-Fan-Glicksberg fixed point theorem (see, e.g., [AB06, Theorem 17.55]) to a restriction of  $\mathcal{E}$  to a suitably chosen domain. Indeed, this theorem applies to correspondences with nonempty, compact, convex domain. Therefore we look for a convex compact subset  $\mathcal{D} \subset D$ , containing  $\mathcal{E}(D)$ , and consider the restriction of  $\mathcal{E}$  on  $\mathcal{D}$ , which we still denote by  $\mathcal{E} : \mathcal{D} \rightarrow \mathcal{D}$ .

Furthermore we need to show that  $\mathcal{E}$  is upper hemicontinuous with non-empty convex values. Lemma A.5 implies the joint continuity of function  $J$  defined in (10), whereas Lemma A.4 provides that  $\mathcal{R}$  is continuous and has nonempty compact values, and therefore by applying the Berge Maximum Theorem (see [AB06, Theorem 17.31]), it is found that the correspondence  $\mathcal{R}^*$  is indeed upper hemicontinuous with nonempty compact values. By linearity and continuity of  $\mathcal{P}(\Omega) \ni P \mapsto \mathbb{E}^P[X] \in D$  so is  $\mathcal{E}$ .

Moreover, by Remark 2,  $\mathcal{R}(m)$  is convex for each  $m$ . Hence by linearity and continuity of  $P \mapsto J(m, P)$  and of  $P \mapsto \mathbb{E}^P[X]$ , it follows that also  $\mathcal{R}^*(m)$  and  $\mathcal{E}(m)$  are convex sets for each  $m \in D$ .

Let  $\chi \in \mathcal{P}^2(\mathbb{R})$  be a given initial law. Let  $\mathcal{Q}$  be a set of the probability measures  $P$  in  $\mathcal{P}(\Omega[A])$ , such that:

- (i)  $X_0 \sim \chi$ ;
- (ii)  $\mathbb{E}^P (|X|_T^*)^2 \leq C$ , where  $C = C(T, c_1, c_\lambda, \chi)$  denotes the constant appearing in equation (33) of Lemma A.1, which is independent of  $P$ ;
- (iii)  $X$  is adapted to a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  and satisfies

$$(14) \quad \mathbb{E}^P \left[ (X_{(t+u) \wedge T} - X_t)^2 \mid \mathcal{F}_t \right] \leq C_1 \delta$$

for  $t \in [0, T]$  and  $u \in [0, \delta]$ , with  $C_1 = \bar{C}(c_1, c_\lambda, \sigma)$  defined in (38) independently of  $P$ .

We need to prove that  $\mathcal{Q}$  is a non empty, compact and convex set.

First,  $\mathcal{Q}$  is convex by construction: consider  $\tilde{P} = \lambda P_1 + (1 - \lambda)P_2$  for  $\lambda \in [0, 1]$  and  $P_1, P_2 \in \mathcal{Q}$  and corresponding filtrations  $\mathbb{F}^1$  and  $\mathbb{F}^2$  as in condition (iii) above. Conditions (i) and (ii) are easily satisfied by  $\tilde{P}$  since the initial distribution  $\chi$  is the same for all the probabilities and the constant  $C$  does not depend on them. Condition (iii) for  $\tilde{P}$  also holds with the same constant  $C_1$  as in (14) and the filtration  $\tilde{\mathbb{F}} = \mathbb{F}^1 \wedge \mathbb{F}^2$ . Furthermore,  $\mathcal{Q}$  is compact, since it satisfies the sufficient criterion for tightness [Whi07, Lemma 3.11]. Indeed since the constant  $C_1$  in (14) is independent of  $P$ , it suffices to choose (in the notation of [Whi07, Lemma 3.11])  $Z(\delta) = C_1 \delta$ , and the result follows. We can now define  $\mathcal{D}$  as follows

$$\mathcal{D} = \{n \in D : \text{there exists } P \in \mathcal{Q} \text{ such that } n = \mathbb{E}^P[X]\} \subset D.$$

Since  $\mathcal{Q}$  is compact and convex and  $P \mapsto \mathbb{E}^P[X]$  is a continuous, linear function,  $\mathcal{D}$  turns out to be a convex, compact set as requested in Kakutani-Fan-Glicksberg fixed point theorem.

In order to show that the range of  $\mathcal{E}$  is contained in  $\mathcal{D}$  we prove that  $\mathcal{R}(m) \subset \mathcal{Q}$  for each  $m \in D$ , so that  $\mathcal{E}(m) \in \mathcal{D}$  for all  $m \in D$  and therefore  $\mathcal{E}(D) \subset \mathcal{D}$ . Let  $P \in \mathcal{R}(m)$ , then it satisfies conditions (i) and (ii) by construction (see Definition 2.1 and Lemma A.1). The validity of condition (iii) can be proved arguing as in Proposition A.2. Indeed, using the same notation therein, we have that for each  $u \in [0, \delta]$

$$\mathbb{E}^P \left[ (X_{(t+u) \wedge T} - X_t)^2 \mid \mathcal{F}_t \right] \leq \bar{C}(c_1, c_\lambda, \sigma)u \leq \bar{C}(c_1, c_\lambda, \sigma)\delta,$$

giving the same bound as in (14) with constant  $\bar{C}(c_1, c_\lambda, \sigma)$ , which does not depend on  $P$ . Note that since  $\mathcal{R}(m)$  is nonempty (see Remark 2) then also  $\mathcal{Q}$  is so.

Since all the hypotheses of the Kakutani-Fan-Glicksberg fixed point theorem are satisfied, we can conclude that there exists a fixed point for the correspondence  $\mathcal{E}$ , that is a relaxed MFG solution  $\square$

The following theorem extends to our setting the results in [Lac15, Theorem 3.7], showing that for any element  $P \in \mathcal{P}(m)$ , for  $m \in D$ , there exists a Markovian control with a lower cost than  $P$ . In our setting we need to assume the integrability property (12) of the characteristic function of  $X_t$  under  $P$ . The reason for this additional assumption will be clear from the proof. In few words, it guarantees the existence of a regular and continuous version of the conditional distribution of  $X$  given the control.

**Theorem 3.2.** *Under Assumption A, let  $P$  be a relaxed MFG solution fulfilling (12). Then there exists a Markovian MFG solution.*

*Moreover, if also Assumption C holds, there exists a strict Markovian MFG solution.*

*Proof.* Let  $P$  be a relaxed MFG solution satisfying (12). In order to show the existence of a Markovian MFG solution we exhibit a (possibly different) probability measure  $P^* \in \mathcal{R}(m)$ , for some  $m \in D$ , such that the following holds:

**Property MP.** (MP.1)  $J(m, P^*) \leq J(m, P)$ ;



(MP.2)  $P^* \circ X_t^{-1} = P \circ X_t^{-1}$  for all  $t \in [0, T]$ ;

(MP.3)  $P^*(\Gamma(dt, d\alpha) = \hat{\Gamma}(t, X_{t-})(d\alpha)dt) = 1$  for a measurable function  $\hat{\Gamma}: [0, T] \times \mathbb{R} \rightarrow \mathcal{V}$ .

Condition (MP.1) implies that under this new probability measure  $P^*$  the expected cost is minimized with respect to all the admissible laws, i.e.  $P^* \in \mathcal{R}^*(m)$ , whereas (MP.2) assures that the fixed point condition is still verified under  $P^*$  and therefore also  $P^*$  is a relaxed MFG solution. Condition (MP.3) is the Markovian property we want for  $P^*$ .

By Definition 2 and Lemma 2.1,  $P$  being a relaxed MFG solution means that there exists a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  such that

$$dX_t = b(t, X_t, m(t))dt + \sigma(t, X_t)dW_t + \lambda(m(t-)) \int_A \alpha \tilde{N}(dt, d\alpha)$$

where  $N$  is a Poisson measure with compensator  $\mathcal{L} \times \Gamma_t$ . Let  $\eta$  be the probability measure defined on the product set  $[0, T] \times \mathbb{R} \times A$  by

$$\eta(I) = \frac{1}{T} \mathbb{E}^P \left[ \int_{[0, T] \times A} \mathbb{1}_I(t, X_t, \alpha) \Gamma_t(d\alpha) dt \right] \quad \text{for each measurable } I \subset [0, T] \times \mathbb{R} \times A,$$

and let  $\hat{\Gamma}$  be its marginal distribution given  $(t, x)$ , i.e.  $\hat{\Gamma}(t, x)(\cdot) = \eta_{(t, x)}(\cdot)$ , where  $\eta(dt, dx, d\alpha) = \eta_{1,2}(dt, dx) \eta_{(t, x)}(d\alpha)$ . Note that  $\hat{\Gamma}$  is measurable, as requested in condition (MP.3), just by construction.

Since for all  $h: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\phi: [0, T] \times \mathbb{R} \times A \rightarrow \mathbb{R}$ , measurable, bounded functions, it holds that

$$\begin{aligned} \mathbb{E}^P \left[ \int_0^T h(t, X_t) \int_A \phi(t, X_t, \alpha) \hat{\Gamma}(t, X_t)(d\alpha) dt \right] &= T \int_{[0, T] \times \mathbb{R}} h(t, x) \int_A \phi(t, x, \alpha) \hat{\Gamma}(t, x)(d\alpha) \eta_{1,2}(dt, dx) \\ &= T \int_{[0, T] \times \mathbb{R} \times A} h(t, x) \phi(t, x, \alpha) \eta(dt, dx, d\alpha) \\ &= \mathbb{E}^P \left[ \int_0^T h(t, X_t) \int_A \phi(t, X_t, \alpha) \Gamma_t(d\alpha) dt \right] \end{aligned}$$

and thanks to [BS13, Lemma 5.2], we have that

$$(15) \quad \int_A \phi(t, X_t, \alpha) \hat{\Gamma}(t, X_t)(d\alpha) = \mathbb{E}^P \left[ \int_A \phi(t, X_t, \alpha) \Gamma_t(d\alpha) \middle| X_t \right] = \mathbb{E}^P \left[ \int_A \phi(t, X_t, \alpha) \Gamma_t(d\alpha) \middle| X_{t-} \right] \quad \text{a.s.}$$

Now, consider the characteristic function of  $\eta$ , i.e.

$$\varphi_\eta(\zeta) := \int_{\mathcal{Z}} e^{i\zeta \cdot z} \eta(dz), \quad \zeta := (\zeta_t, \zeta_x, \zeta_\alpha) \in \mathbb{R}^3,$$

where  $z := (t, x, \alpha)$  and  $\mathcal{Z} = [0, T] \times \mathbb{R} \times A$ . Since  $[0, T] \times A$  is compact and  $\sup_t |\varphi_{X_t}(\zeta_x)| \in L^1(\mathbb{R})$  by assumption, we have that

$$\begin{aligned} |\varphi_\eta(\zeta)| &= \left| \frac{1}{T} \mathbb{E}^P \left[ \int_{[0, T] \times A} e^{i\zeta \cdot (t, X_t, \alpha)} \Gamma_t(d\alpha) dt \right] \right| \\ &\leq \frac{C}{T} \int_{[0, T] \times A} |\mathbb{E}^P [e^{i\zeta_x \cdot X_t}]| \Gamma_t(d\alpha) dt \\ &\leq C \sup_t |\varphi_{X_t}(\zeta_x)| =: \kappa(\zeta_x) \in L^1(\mathbb{R}). \end{aligned}$$

It follows that  $\varphi_\eta(\zeta)$  is dominated, uniformly in  $(t, \alpha)$ , by the function  $\kappa(\zeta_x)$  belonging to  $L^1(\mathbb{R})$ . Hence we can apply [Zab79, Theorem 3.2], yielding that the joint measure  $\eta$  admits a regular and continuous (in the sense of weak convergence) version of the conditional probability  $\hat{\Gamma}(t, x)(d\alpha)$ .

By a similar result established in [BC09, Theorem 2] (see Appendix B), there exists a probability space  $(\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{Q})$ , a Brownian motion  $B$ , a Poisson process  $\hat{N}$  with compensator  $\mathcal{L} \times \hat{\Gamma}$ , and an adapted process  $Z$  satisfying

$$Z_t = X_0 + \int_0^t b(s, Z_s, m(s))ds + \int_0^t \sigma(s, Z_s)dB_s + \int_{[0,t] \times A} \alpha \tilde{N}(dt, d\alpha)$$

such that its marginal distributions mimic that of  $X$ , i.e.  $\hat{Q} \circ Z_t^{-1} = P \circ X_t^{-1}$  for all  $t \in [0, T]$ .

Let  $P^* := \hat{Q} \circ (\hat{\Gamma}(t, Z_t)dt, Z)^{-1}$ . By [BC09, Theorem 2] (see Appendix B)  $P^* \in \mathcal{R}(m)$  and it satisfies conditions (MP.2) and (MP.3) by construction. Moreover we have that

$$\begin{aligned} J(m, P^*) &= \mathbb{E}^{\hat{Q}} \left[ \int_{[0,T] \times A} f(t, Z_t, m(t), \alpha) \hat{\Gamma}(t, Z_t)(d\alpha)dt + g(Z_T, m(T)) \right] \\ &\stackrel{(a)}{=} \mathbb{E}^P \left[ \int_{[0,T] \times A} f(t, X_t, m(t), \alpha) \hat{\Gamma}(t, X_t)(d\alpha)dt + g(X_T, m(T)) \right] \\ &\stackrel{(b)}{=} \mathbb{E}^P \left[ \int_{[0,T] \times A} \mathbb{E}^P \left[ f(t, X_t, m(t), \alpha) \Gamma_t(d\alpha) \middle| X_{t-} \right] dt + g(X_T, m(T)) \right] \\ &\stackrel{(c)}{=} \mathbb{E}^P \left[ \int_{[0,T] \times A} f(t, X_t, m(t), \alpha) \Gamma_t(d\alpha)dt + g(X_T, m(T)) \right] \\ &= J(m, P), \end{aligned}$$

where equality (a) follows from the equivalent distribution of the processes involved, i.e.  $\hat{Q} \circ Z_t^{-1} = P \circ X_t^{-1}$  for all  $t \in [0, T]$ , and from  $\hat{Q} \circ (\Gamma_t(da)dt, X_t)^{-1} \equiv P$ , equality (b) is provided by (15) and equality (c) is just the tower property of conditional expectations. Therefore  $P^*$  satisfies condition (MP.1) as well, so that the proof of the first part of the theorem is complete.

For the existence of a strict Markovian MFG solution, assume that the set  $K(t, x, m)$  defined in (11) is convex for all  $(t, x, m) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . Hence by applying the same arguments as in the second part of the proof of Theorem 3.7 in [Lac15], we get the existence of a measurable function  $\hat{\gamma} : [0, T] \times \mathbb{R} \rightarrow A$  such that  $\hat{\Gamma}(t, x)(d\alpha) = \delta_{\hat{\gamma}(t, x)}(d\alpha)$ .  $\square$

#### 4. APPLICATION: A TOY MODEL FOR AN ILLIQUID INTER-BANK MARKET

In this section we consider a symmetric  $N$ -player game with controlled jumps fitting the general framework studied in the previous sections.<sup>1</sup> The model shows the economic relevance of the general MFG considered in the previous section and, furthermore, it is simple enough to allow for explicit computations of the equilibrium in  $N$ -player case as well as when  $N \rightarrow \infty$ . The convergence to the MFG solution is also studied. The model is in our opinion an interesting variation of the systemic risk model proposed by Carmona and coauthors in [CFS15], parsimoniously modified for including some illiquidity phenomenon.

More precisely, we consider  $N > 0$  banks which lend to and borrow from a central bank in an interbank borrowing market. The aim of each bank is to keep its monetary reserves away from

<sup>1</sup>We refer to Remark 1 on how to deal with the unbounded coefficient case.

critical levels. After the financial crisis of 2008, banks are required by international regulation to store an adequate amount of liquid assets, cash, to manage possible market stress. At the same time, holding too much cash is costly for banks because of its low return. We will use the average monetary reserves of the system as the benchmark for the reserves' level of each bank and so the cost function will penalize every deviation from this mean value as in [CFS15].

The main difference with the model in [CFS15] is that therein the banks can control their reserves continuously over time, i.e. they can choose a rate at which lend or borrow money, while in the present paper the interbank market is illiquid. This means that the banks can borrow or lend money only at some exogenously given instants, that will be modelled as jump times of a Poisson process with a certain intensity  $\lambda > 0$ . The intensity can be viewed as an health indicator of the whole system: for instance, when the intensity is low, hence the probability of being able to control the reserves in the next instant is also low, the system is becoming very illiquid. This kind of situation typically arises during financial crises.

Now, we turn to the mathematical description of the model. Let  $X^i = (X_t^i)_{t \in [0, T]}$  denote the monetary log-reserves of bank  $i$ , whose evolution is given by

$$(16) \quad dX_t^i = \frac{a}{N} \sum_{j=1}^N (X_t^j - X_t^i) dt + \sigma dW_t^i + \lambda \gamma_t^i dP_t^i, \quad i = 1, \dots, N,$$

where  $\lambda > 0$  is some constant,  $(W^1, \dots, W^N)$  is an  $N$ -dimensional Brownian motion and  $(P^1, \dots, P^N)$  is an  $N$ -dimensional Poisson process, each component with a unitary intensity. We assume that the system starts at time  $t = 0$  from i.i.d. random variables  $X_0^i = \xi^i$  such that  $\mathbb{E}[\xi^i] = 0$  for all  $i = 1, \dots, N$ , and that initial conditions, Brownian motions and Poisson processes are all independent. Notice that we use  $P^i$  (instead of  $N^i$  as in the previous section) to denote the jump component, to avoid confusion with  $N$  denoting the number of players.

The main difference with respect to [CFS15] is that the control  $\gamma_t^i$  appears only in the jump component, hence the bank  $i$  cannot change its reserves continuously over time but only at the jump times of the Poisson process  $P^i$ . Notice that while the banks can borrow/lend money at different times, being the  $P^i$ 's independent, the parameter  $\lambda$  measuring the intensity of the jump component is the same for each bank.

We denote by  $\bar{X}_t$  the empirical mean of the monetary log-reserves, i.e.

$$\bar{X}_t = \frac{1}{N} \sum_{i=1}^N X_t^i,$$

hence the dynamics of bank  $i$  reserves can be rewritten in the mean field form as

$$dX_t^i = a(\bar{X}_t - X_t^i) dt + \sigma dW_t^i + \lambda \gamma_t^i dP_t^i, \quad i = 1, \dots, N,$$

and the dynamics of the average state,  $\bar{X}_t$ , can be expressed as

$$d\bar{X}_t = \frac{1}{N} \sum_{k=1}^N dX_t^k = \frac{\lambda}{N} \sum_{k=1}^N \gamma_t^k dt + \frac{\sigma}{N} \sum_{k=1}^N dW_t^k + \frac{\lambda}{N} \sum_{k=1}^N \gamma_t^k d\tilde{P}_t^k.$$

The dynamics of the monetary reserves are coupled through their drifts by means of the average state of the system as in [CFS15]. Let  $X = (X^1, \dots, X^N)$  and  $\gamma = (\gamma^1, \dots, \gamma^N)$ . Bank  $i = 1, \dots, N$  controls its rate of lending/borrowing at time  $t$ ,  $\gamma_t^i$ , in order to minimize a cost functional  $J^i$ , where

$$J^i(\gamma) = J^i(\gamma^1, \dots, \gamma^N) = \mathbb{E} \left[ \int_0^T \lambda f^i(X_t, \gamma_t^i) dt + g^i(X_T) \right],$$

where  $f^i$  and  $g^i$  are quadratic functions as in [CFS15], namely the running cost function  $f^i: \mathbb{R}^N \times A \rightarrow \mathbb{R}$  is given by

$$f(x, \gamma^i) = \frac{1}{2}(\gamma^i)^2 - \theta \gamma^i(\bar{x} - x^i) + \frac{\varepsilon}{2}(\bar{x} - x^i)^2,$$

where  $A$  is a fixed subset of  $\mathbb{R}$ ,  $\bar{x} = \frac{1}{N} \sum_{i=1}^N x^i$  and the terminal cost function  $g^i: \mathbb{R}^N \rightarrow \mathbb{R}$  is

$$g^i(x) = \frac{c}{2}(\bar{x} - x^i)^2.$$

Note that both cost functions of player  $i$  depend on the other players' strategies through the mean, i.e.  $f^i(x, \gamma^i) = f(\bar{x}, x^i, \gamma^i)$  and  $g^i(x) = g(\bar{x}, x^i)$ . The parameter  $\theta > 0$  is to control the incentive to borrowing or lending: the bank  $i$  will want to borrow (i.e.  $\gamma_t^i > 0$ ) if  $X_t^i$  is smaller than the empirical mean  $\bar{X}_t$  and lend (i.e.  $\gamma_t^i < 0$ ) if  $X_t^i$  is greater than the mean  $\bar{X}_t$ . We have taken the shape of the objective functions from [CFS15] (see therein for more details on the financial interpretation of such cost functions).

The parameters  $\varepsilon$  and  $c$  are strictly greater than 0, so that the quadratic terms  $(\bar{x} - x^i)^2$  in both costs penalize departure from the average. Moreover we assume that

$$\theta^2 \leq \varepsilon,$$

which guarantees the convexity of  $f^i(x, \gamma)$  in  $(x, \gamma)$ . In the next sub-sections, we will compute the Nash equilibria in open-loop as well as closed-loop form (see Remark 4 below). Moreover, we will study the corresponding MFG when  $N \rightarrow \infty$ , whose solution will be strict Markovian. Our approach is heavily based on BSDE and it follows closely the one in [CFS15]. Nonetheless, we give all details at least in the open-loop case for reader's convenience. Finally, we will conclude with some simulations and some financial comments on the model. In what follows we assume that the space of actions,  $A$ , is a closed, compact interval of the real line, i.e.  $A = [a_0, a_1]$  with  $-\infty < a_0 < a_1 < \infty$ .

**4.1. The open-loop problem.** We are searching for a Nash equilibrium among all admissible open-loop strategies  $\gamma_t = \{\gamma_t^i, i = 1, \dots, N\}$ , that are predictable processes with values in  $A$  satisfying the integrability condition  $\mathbb{E}[\int_0^T |\gamma_t^i| dt] < \infty$  for all  $i = 1, \dots, N$ . We denote the set of all these admissible controls by  $\mathcal{A}$ .

For the problem of the  $i$ -th bank, we consider the Hamiltonian

$$H^i(t, x, \gamma, y^i, q^i, r^i) : [0, T] \times \mathbb{R}^N \times A \times \mathbb{R}^N \times \mathbb{R}^{N \times N} \times \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$$

defined by

$$\begin{aligned} H^i(t, x, \gamma, y^i, q^i, r^i) &= \lambda f^i(x, \gamma) + (a(\bar{x} - x) + \lambda \gamma) \cdot y^i + \sigma \text{tr}(q^i) + \lambda \gamma \cdot \text{diag}(r^i) \\ (17) \quad &= \lambda \left( \frac{(\gamma^i)^2}{2} - \theta(\bar{x} - x^i)\gamma^i + \frac{\varepsilon}{2}(\bar{x} - x^i)^2 \right) + \sum_{k=1}^N [a(\bar{x} - x^k) + \lambda \gamma^k] y^{i,k} \\ &\quad + \sigma \sum_{k=1}^N q^{i,k,k} + \lambda \sum_{k=1}^N \gamma^k r^{i,k,k}. \end{aligned}$$

The adjoint processes  $Y_t^i = \{Y_t^{i,k} : k = 1, \dots, N\}$ ,  $Q_t^i = \{Q_t^{i,k,j} : k, j = 1, \dots, N\}$  and  $R_t^i = \{R_t^{i,k,j} : k, j = 1, \dots, N\}$  are defined as the solutions of the following BSDEs with jumps (see, e.g., [Del13,

Theorem 3.1.1])

$$(18) \quad \begin{cases} dY_t^{i,k} = -\frac{\partial H^i(t, X_t, \gamma_t, Y_t^i, Q_t^i, R_t^i)}{\partial x^k} dt + \sum_{j=1}^N Q_t^{i,k,j} dW_t^j + \sum_{j=1}^N R_t^{i,k,j} d\tilde{P}_t^j \\ Y_T^{i,k} = \frac{\partial g^i}{\partial x^k}(X_T), \end{cases}$$

for  $k = 1, \dots, N$ .

In order to find a candidate for the optimal control  $\hat{\gamma}^i$ , it suffices to minimize the Hamiltonian  $H^i$  with respect to  $\gamma^i$ , leading to

$$(19) \quad \hat{\gamma}^i = \theta(\bar{x} - x^i) - (y^{i,i} + r^{i,i,i}).$$

In order to prove that  $\hat{\gamma} = (\hat{\gamma}^1, \dots, \hat{\gamma}^N)$  is a Nash equilibrium we show that when the other players  $j \neq i$  are following  $\hat{\gamma}^j$ , then  $\hat{\gamma}^i$  is the best response of player  $i$ . To do that, we need to solve the BSDE with jumps (18). It is natural to consider the ansatz

$$(20) \quad Y_t^{i,k} = \left( \frac{1}{N} - \delta_{i,k} \right) (\bar{X}_t - X_t^i) \phi_t,$$

where  $\delta_{i,j}$  is the Kronecker delta and  $\phi$  is a deterministic scalar function of class  $C^1$ , satisfying the terminal condition  $\phi_T = c$ , so that  $Y_T^{i,k} = \frac{\partial g^i}{\partial x^k}(X_T) = c(\frac{1}{N} - \delta_{i,k})(\bar{X}_T - X_T^i)$ . Differentiating the ansatz, we have that  $Y^{i,k}$  solves the following SDE:

$$(21) \quad \begin{aligned} dY_t^{i,k} &= \left( \frac{1}{N} - \delta_{i,k} \right) d(\bar{X}_t - X_t^i) \phi_t + \left( \frac{1}{N} - \delta_{i,k} \right) (\bar{X}_t - X_t^i) \dot{\phi}_t dt \\ &= \left( \frac{1}{N} - \delta_{i,k} \right) [\lambda \phi_t (\bar{\gamma}_t - \gamma_t^i) + (\dot{\phi}_t - a \phi_t) (\bar{X}_t - X_t^i)] dt + \left( \frac{1}{N} - \delta_{i,k} \right) \phi_t \sigma \left( \frac{1}{N} \sum_{j=1}^N dW_t^j - dW_t^i \right) \\ &\quad + \left( \frac{1}{N} - \delta_{i,k} \right) \phi_t \lambda \left( \frac{1}{N} \sum_{j=1}^N \gamma_t^j d\tilde{P}_t^j - \gamma_t^i d\tilde{P}_t^i \right), \end{aligned}$$

where  $\bar{\gamma}_t = \frac{1}{N} \sum_{k=1}^N \gamma_t^k$  denotes the average value of all control processes. Comparing (18) and (21) under the ansatz (20) yields

$$(22) \quad Q_t^{i,k,j} = \sigma \left( \frac{1}{N} - \delta_{i,k} \right) \left( \frac{1}{N} - \delta_{i,j} \right) \phi_t,$$

$$(23) \quad R_t^{i,k,j} = \lambda \left( \frac{1}{N} - \delta_{i,k} \right) \left( \frac{1}{N} - \delta_{i,j} \right) \phi_t \gamma_t^j,$$

for all  $k, j = 1, \dots, N$ .

Moreover, by (20) and (23), it follows that the candidate  $\hat{\gamma}^i$  given in (19) solves

$$\hat{\gamma}_t^i = \theta(\bar{X}_{t-} - X_{t-}^i) - \left( \frac{1}{N} - 1 \right) \phi_t (\bar{X}_{t-} - X_{t-}^i) - \lambda \left( \frac{1}{N} - 1 \right)^2 \phi_{t-} \hat{\gamma}_t^i,$$

and therefore the candidate optimal best response  $\hat{\gamma}^i$  turns out to be

$$(24) \quad \hat{\gamma}_t^i = \frac{\theta + (1 - \frac{1}{N}) \phi_t}{1 + \lambda (1 - \frac{1}{N})^2 \phi_t} (\bar{X}_{t-} - X_{t-}^i).$$

It should be noted that even if in principle we were looking for an open-loop optimal strategy, it turned out to have a closed-loop structure, since  $\hat{\gamma}_t^i = \hat{\gamma}^i(t, X_{t-})$ .

So far we have not yet taken into account the constraint that requires the strategies to take values in  $A$ , i.e.  $\gamma_t^i \in [a_0, a_1]$  for all  $t \in [0, T]$ . Therefore  $\hat{\gamma}^i$  has to be modified as follows

$$\hat{\gamma}_t^{i,A} := \begin{cases} a_0 & \text{if } \hat{\gamma}_t^i < a_0 \\ \hat{\gamma}_t^i & \text{if } \hat{\gamma}_t^i \in [a_0, a_1] \\ a_1 & \text{if } \hat{\gamma}_t^i > a_1 \end{cases} \quad t \in [0, T].$$

To complete the description of  $\hat{\gamma}^i$ , we need to provide a characterization of the function  $\phi$ . From the definition of the Hamiltonian  $H^i$  in (17) it follows that the drift coefficient in the SDE (18) for  $Y$  as adjoint process is given by

$$-\frac{\partial H^i(t, x, \gamma, y^i, q^i, r^i)}{\partial x^k} = \lambda \theta \left( \frac{1}{N} - \delta_{i,k} \right) \gamma^i - \lambda \varepsilon \left( \frac{1}{N} - \delta_{i,k} \right) (\bar{x} - x^i) - \frac{a}{N} \sum_{j=1}^N (y^{i,j} - y^{i,k}),$$

and therefore, under the ansatz (20) and the related implications, equation (18) becomes

$$(25) \quad dY_t^{i,k} = \left( \frac{1}{N} - \delta_{i,k} \right) \left[ \frac{\lambda \theta^2 + \lambda \theta \left( 1 - \frac{1}{N} \right) \phi_t}{1 + \lambda \left( 1 - \frac{1}{N} \right)^2 \phi_t} - \lambda \varepsilon + a \phi_t \right] (\bar{X}_t - X_t^i) dt \\ + \sum_{j=1}^N \left( Q_t^{i,k,j} dW_t^j + R_t^{i,j,k} d\tilde{P}_t^j \right).$$

Since both equations (21) and (26) hold simultaneously, we have that the following equality holds

$$\dot{\phi} - a\phi - \frac{\lambda \theta + \lambda \left( 1 - \frac{1}{N} \right) \phi}{1 + \lambda \left( 1 - \frac{1}{N} \right)^2 \phi} \phi = \frac{\lambda \theta^2 + \lambda \theta \left( 1 - \frac{1}{N} \right) \phi}{1 + \lambda \left( 1 - \frac{1}{N} \right)^2 \phi} - \lambda \varepsilon + a\phi_t$$

and this implies that  $\phi_t$  must solve the ODE

$$(26) \quad \left( 1 + \lambda \left( 1 - \frac{1}{N} \right)^2 \phi_t \right) \dot{\phi}_t = \left[ 1 + 2a \left( 1 - \frac{1}{N} \right) \right] \lambda \left( 1 - \frac{1}{N} \right) \phi_t^2 \\ + \left[ \lambda \theta \left( 2 - \frac{1}{N} \right) - \varepsilon \lambda^2 \left( 1 - \frac{1}{N} \right)^2 + 2a \right] \phi_t + \lambda (\theta^2 - \varepsilon),$$

with terminal condition  $\phi_T = c$ . For more details on how such an ODE can be solved at least in implicit form we refer the reader to Appendix C.

*Remark 4.* In the closed-loop case, each player has complete information of the states of the other participants, and therefore his best reply is chosen among all Markovian strategies of the form  $\gamma^i(t, X_t)$ ,  $t \in [0, T]$ , for some function  $\gamma_t^i(t, x)$  with values in  $A$ . Following the same approach as before, based on the Pontryaguin maximum principle and BSDEs with jumps, we obtain a Nash equilibrium in closed-loop form  $\hat{\gamma}^{i,A}$  given by

$$\hat{\gamma}_t^{i,A} := \begin{cases} a_0 & \text{if } \hat{\gamma}_t^i < a_0 \\ \hat{\gamma}_t^i & \text{if } \hat{\gamma}_t^i \in [a_0, a_1] \\ a_1 & \text{if } \hat{\gamma}_t^i > a_1 \end{cases} \quad t \in [0, T],$$

where

$$(27) \quad \hat{\gamma}_t^i = \frac{\theta + \left( 1 - \frac{1}{N} \right) \eta_t}{1 + \lambda \left( 1 - \frac{1}{N} \right)^2 \eta_t} (\bar{X}_{t-} - X_{t-}^i)$$

and the function  $\eta_t$  solves the following ODE

$$\begin{aligned} \left(1 + \lambda \left(1 - \frac{1}{N}\right)^2 \eta_t\right)^2 \dot{\eta}_t &= \lambda^2 \left(1 - \frac{1}{N}\right)^2 \left[2a \left(1 - \frac{1}{N}\right)^2 + 1\right] \eta_t^3 \\ &+ \left[4a\lambda \left(1 - \frac{1}{N}\right)^2 - \lambda^3 \varepsilon \left(1 - \frac{1}{N}\right)^4 + \lambda \left(1 - \frac{1}{N^2}\right) + 2\lambda^2 \theta \left(\frac{N^3 - 2N^2 + 2N - 1}{N^3}\right)\right] \eta_t^2 \\ &+ \left[2a + 2\lambda\theta + \lambda^2 \theta^2 \left(\frac{N^2 - 2N + 2}{N^2}\right) - 2\lambda^2 \left(1 - \frac{1}{N}\right)^2 \varepsilon\right] \eta_t + \lambda(\theta^2 - \varepsilon), \end{aligned}$$

with the same terminal condition as before, i.e.  $\eta_T = c$ . We do not develop further on this since a full comparison between open and closed loop equilibria goes beyond the scope of this section, which is more focused on the differences between our model and [CFS15].

**4.2. Approximate Nash equilibria.** We conclude the theoretical study of our model by computing the solution of the MFG obtained from the  $N$ -player game as  $N \rightarrow \infty$ . In particular, we will see that the  $N$ -player Nash equilibria computed in the previous sections tend towards the MFG solution.

Let  $m \in D$  be a given càdlàg function, which represents a candidate for the limit of  $\mathbb{E}[\bar{X}_t]$  when  $N \rightarrow \infty$ . Consider the following one-player minimization problem

$$\inf_{\gamma} \mathbb{E} \left[ \int_0^T \lambda \left( \frac{\gamma_t^2}{2} - \theta \gamma_t (m(t) - X_t) + \frac{\varepsilon}{2} (m(t) - X_t)^2 \right) dt + \frac{c}{2} (m(T) - X_T)^2 \right]$$

subject to the dynamics

$$(28) \quad dX_t = [a(m(t) - X_t) + \lambda \gamma_t] dt + \sigma dW_t + \lambda \gamma_t d\tilde{P}_t, \quad X_0 = x_0,$$

where  $W$  is a standard Brownian motion and  $P$  a standard Poisson process, i.e. with unitary intensity. The processes  $W$  and  $P$  are assumed to be independent. Then, for a given càdlàg function  $m \in D$ , the optimal control  $\hat{\gamma}$  is chosen in the class of admissible controls, which are predictable processes  $\gamma$  with values in the compact interval  $A$  and such that  $\mathbb{E}[\int_0^T |\gamma_t| dt] < \infty$ . Moreover, in order for  $\hat{\gamma}$  to be a MFG solution, the mean-field condition has to be satisfied, i.e.  $\mathbb{E}[X_t^{\hat{\gamma}}] = m(t)$  for a.e.  $t \in [0, T]$ .

As in the  $N$ -player case, we solve the problem via the Pontryagin maximum principle. In this case, the Hamiltonian is given by

$$H(t, x, y, q, r, \gamma) = \lambda \left( \frac{\gamma^2}{2} - \theta \gamma (m(t) - x) + \frac{\varepsilon}{2} (m(t) - x)^2 \right) + [a(m(t) - x) + \lambda \gamma] y + \sigma q + \lambda \gamma r$$

and the first order condition implies that the candidate optimal strategy is

$$\hat{\gamma} = \theta(m(t) - x) - (y + r).$$

The corresponding adjoint forward-backward equations are as follows:

- SDE for the forward process  $X$ :

$$(29) \quad \begin{cases} dX_t = [(a + \lambda\theta)(m(t) - X_t) - \lambda(Y_t + R_t)] dt + \sigma dW_t + \lambda[\theta(m(t-) - X_{t-}) - (Y_{t-} + R_{t-})] d\tilde{P}_t \\ X_0 = x_0 \end{cases}$$

- BSDE for the adjoint processes  $Y, Q, R$ :<sup>2</sup>

$$(30) \quad \begin{cases} dY_t = [(a + \lambda\theta)Y_t + \lambda\theta R_t + \lambda(\varepsilon - \theta^2)(m(t) - X_t)] dt + Q_t dW_t + R_{t-} d\tilde{P}_t \\ Y_T = c(X_T - m(T)) \end{cases}$$

Note that this time the optimization problem is one-dimensional and therefore  $Y$ ,  $Q$  and  $R$  are real-valued stochastic processes.

Since equations (29)-(30) are linear we can firstly solve for the expectations  $\mathbb{E}[X_t]$  and  $\mathbb{E}[Y_t]$ . By taking expected value in both sides of equations (29) and (30) and using the martingale property of the integrals with respect to Brownian motion and compensated Poisson process, we have

$$d\mathbb{E}[X_t] = [(a + \lambda\theta)(m(t) - \mathbb{E}[X_t]) - \lambda(\mathbb{E}[Y_t] + \mathbb{E}[R_t])] dt.$$

which becomes

$$dm(t) = -\lambda(\mathbb{E}[Y_t] + \mathbb{E}[R_t]) dt,$$

under the fixed-point assumption  $m(t) = \mathbb{E}[X_t]$ .

In order to solve the BSDE with jumps (30), we make the same ansatz as before:

$$Y_t = -\phi_t(m(t) - X_t), \quad Q_t = \sigma\phi_t, \quad R_t = \lambda \frac{\theta + \phi_t}{1 + \lambda\phi_t} \phi_t(m(t-) - X_{t-})$$

for some deterministic function  $\phi$  of class  $C^1$  with final value  $\eta_T = c$ , so that  $Y_T = c(X_T - m(T))$  as required by the BSDE (30). Notice that these processes can be obtained by those in (20)-(22)-(23) by letting  $N \rightarrow \infty$ .

If we plug the ansatz in the BSDE (30), we find that the process  $Y_t$  solves the following SDE

$$(31) \quad dY_t = \left( -a\phi_t + \lambda\varepsilon - \lambda\theta \frac{\theta + \phi_t}{1 + \lambda\phi_t} \right) dt + Q_t dW_t + R_{t-} d\tilde{P}_t$$

and, at the same time, by differentiating  $Y_t$ , we have that

$$\begin{aligned} dY_t &= \left( -\dot{\phi}_t + a\phi_t + \lambda \frac{\theta + \phi_t}{1 + \lambda\phi_t} \phi_t + \lambda^2(\mathbb{E}[Y_t] + \mathbb{E}[R_t]) \right) (m(t) - X_t) dt \\ &\quad + \sigma\eta_t dW_t + \lambda \frac{\theta + \phi_t}{1 + \lambda\phi_t} \phi_t(m(t-) - X_{t-}) d\tilde{P}_t \\ &= \left( -\dot{\phi}_t + a\phi_t + \lambda \frac{\theta + \phi_t}{1 + \lambda\phi_t} \phi_t \right) (m(t) - X_t) dt + \sigma\eta_t dW_t + \lambda \frac{\theta + \phi_t}{1 + \lambda\phi_t} \phi_t(m(t-) - X_{t-}) d\tilde{P}_t, \end{aligned}$$

where once again we used the identity  $m(t) = \mathbb{E}[X_t]$  and the equalities  $\mathbb{E}[Y_t] = 0 = \mathbb{E}[R_t]$ , which are due to the fact that both processes are proportional to  $m(t) - \mathbb{E}[X_t]$ . By matching the two SDEs for  $Y$ , we find that  $\phi_t$  solves the following Cauchy problem

$$(32) \quad \begin{cases} (1 + \lambda\phi_t)\dot{\phi}_t = (2a + 1)\lambda\phi_t^2 + (2a + 2\lambda\theta - \lambda^2\varepsilon)\phi_t - \lambda(\varepsilon - \theta^2) \\ \phi(T) = c \end{cases}$$

and therefore the optimal control turns out to be

$$\hat{\gamma}_t^A := \begin{cases} a_0 & \text{if } \hat{\gamma}_t < a_0 \\ \hat{\gamma}_t = \frac{\theta + \phi_t}{1 + \lambda\phi_t} (\mathbb{E}[X_{t-}] - X_{t-}) & \text{if } \hat{\gamma}_t \in [a_0, a_1] \\ a_1 & \text{if } \hat{\gamma}_t > a_1 \end{cases}$$

---

<sup>2</sup>For existence and uniqueness of the solution for this BSDE with jumps we refer again to [Del13, Theorem 3.1.1].



Observe that this can also be obtained as limit for  $N \rightarrow \infty$  of the Nash equilibrium computed before in the  $N$ -player game (see equations (24) and (26)). Figure 1 displays the behaviour of  $\phi$ , solution of the ODE (26), for different values of players' number  $N$ . As  $N$  increases, the graph of  $\phi = \phi(N)$  quickly converges to the solution we found in the game with an infinite number of players, given in equation (32).

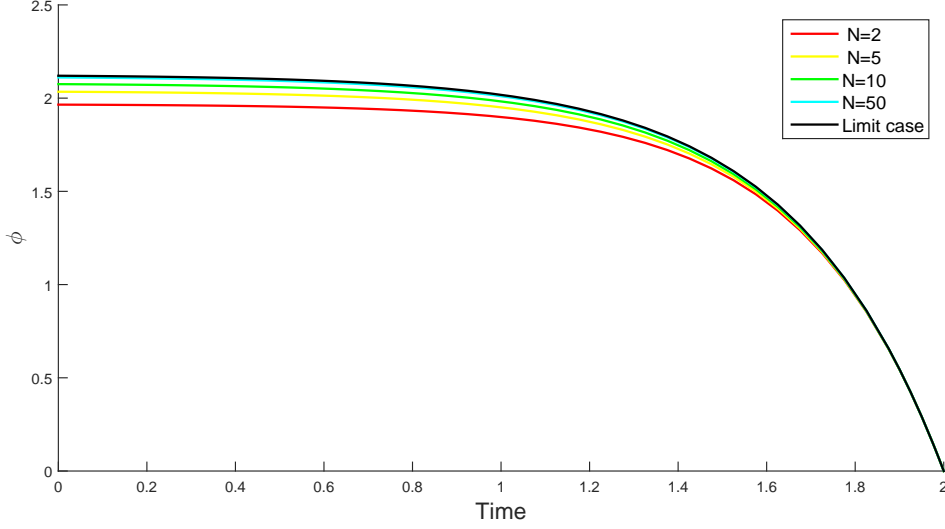


FIGURE 1. Plots of  $\phi$ , solution of the ODE (26) for different values of  $N$ . Model's parameter:  $T = 2$ ,  $a = 1$ ,  $\theta = 1$ ,  $\varepsilon = 10$ ,  $\lambda = 0.7$ ,  $c = 0$

**4.3. Simulations.** Previous computations in Section 4.1 and Remark 4 shows that the open-loop optimal strategies (see equation (24)) and the closed-loop optimal strategy (see equation (27)) have the same form, that is

$$\hat{\gamma}_t^{i,A} := \begin{cases} a_0 & \text{if } \hat{\gamma}_t^i < a_0 \\ \hat{\gamma}_t^i = \frac{\theta + (1 - \frac{1}{N})\phi_t}{1 + \lambda(1 - \frac{1}{N})^2\phi_t} (\bar{X}_{t-} - X_{t-}^i) & \text{if } \hat{\gamma}_t^i \in [a_0, a_1] \\ a_1 & \text{if } \hat{\gamma}_t^i > a_1 \end{cases} \quad t \in [0, T].$$

for some deterministic function  $\phi$  solving two different ODEs in the two cases as in [CFS15]. In this section, we examine the dependence on the parameters of the model of the open-loop strategies and log-reserves at the Nash equilibrium computed before. The same analysis applied to the closed-loop case would lead to the same qualitative conclusions.

Figure 2 shows a typical scenario of our model. It can be observed that the optimal strategy is such that at each jump time for the reserves of a given bank, i.e. when the bank can borrow or lend money, the reserves move closer to the average level  $\bar{X}$  of the reserves in the system. Indeed, since this average is the benchmark for each bank, they try to replicate it. There are two reasons why they do not match exactly. First, reaching  $\bar{X}$  can be too costly. Second, the choice of each bank, say bank 1, at time  $t-$  depends on the difference between its reserve  $X_{t-}^1$  and the average reserves

$\bar{X}_{t-}$  immediately before time  $t$ , with the aim of reducing such difference. But at the same time,  $\bar{X}$  might have a jump at time  $t$ , as a consequence of the jump in the reserves of bank 1. So even if  $X_t^1 = \bar{X}_{t-}$  we could have  $X_t^1 \neq \bar{X}_t$ .

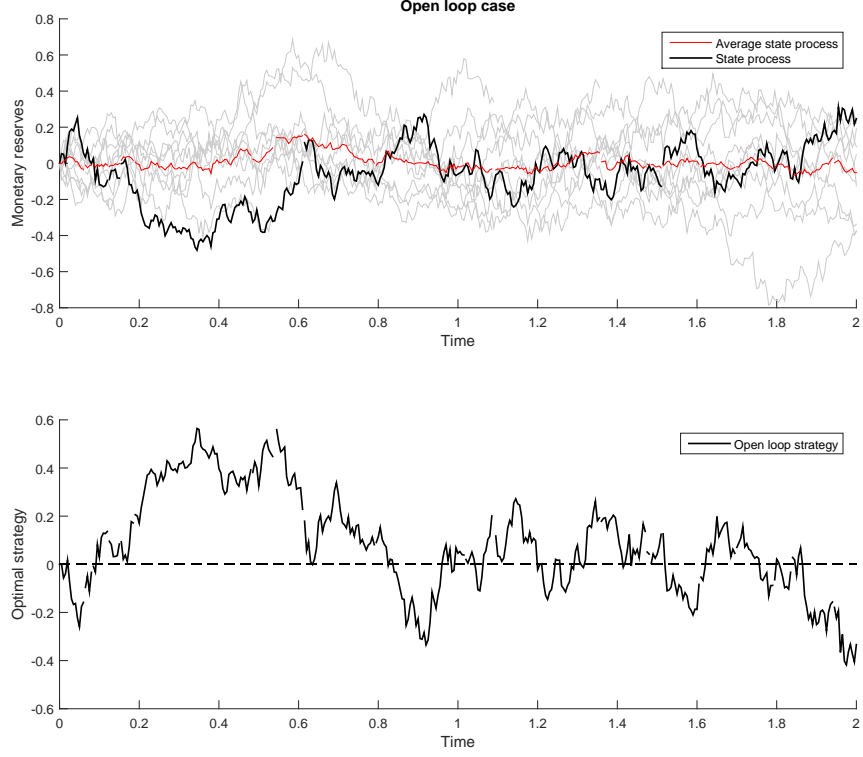
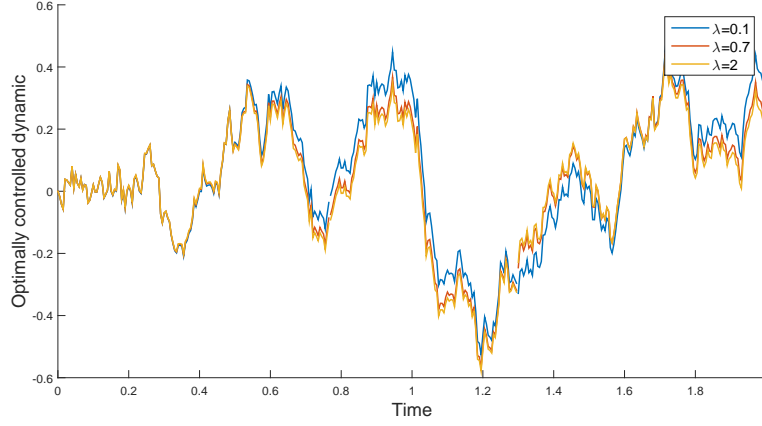


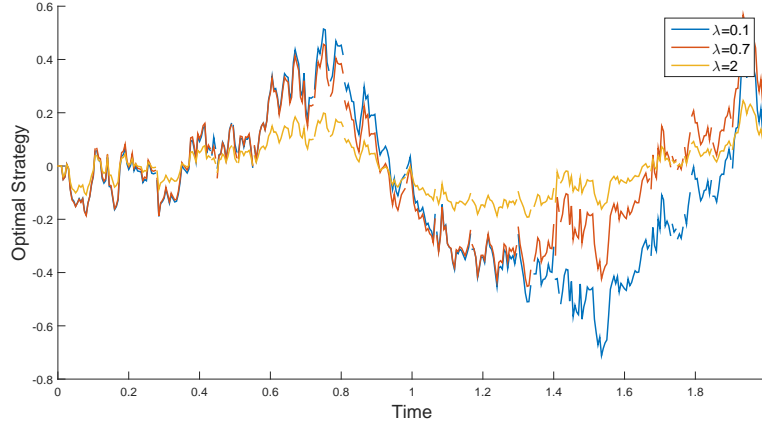
FIGURE 2. Possible scenario. Model's parameters:  $N = 10$ ,  $T = 2$ ,  $a = 1$ ,  $\sigma = 0.8$ ,  $X_0 = 0$ ,  $\theta = 1$ ,  $\varepsilon = 10$ ,  $c = 0$ ,  $\lambda = 0.7$ .

Figure 3 shows how the optimal strategy  $\hat{\gamma}^1$  of the first bank and the corresponding optimal reserves  $\hat{X}^1$  vary with the parameter  $\lambda$ . We recall that  $\lambda$  is an exogenous positive parameter, representing the depth of the interbank lending market: the smaller  $\lambda$ , the more illiquid the market. In Figure 3 we have plotted the evolutions of the optimal strategy of the first bank. These processes are simulated against the same noises, in other terms we used the same paths of the Brownian motions  $W^i$  and of the Poisson processes  $P^i$  in order to focus on the variation due to changes in  $\lambda$  only. Notice that they all jump simultaneously.

Let  $\tau^i$  be a jump time for the Poisson process  $P^i$ . Since in the dynamics of the reserves of bank  $i$  the chosen control  $\gamma_\tau^i$  is multiplied by  $\lambda$ , the actual jump's size in the reserves equals  $\lambda \hat{\gamma}_\tau^i$ . Therefore a small value of  $\lambda$  reduces the effect of  $\gamma^i$ . From a financial point of view,  $\lambda$  taking small values means that a bank has to be more aggressive in the market in order to produce the same changes in the reserves. This can also be seen in Table 1, where we listed the sizes of the jumps in



(A) Optimal log-reserves



(B) Optimal strategy

FIGURE 3. Different values of  $\lambda$ . Model's parameters:  $N = 10$ ,  $T = 2$ ,  $a = 1$ ,  $\sigma = 0.8$ ,  $X_0 = 0$ ,  $\theta = 1$ ,  $\varepsilon = 10$ ,  $c = 0$ .

the dynamic  $X$  and in the optimal strategy, respectively. More precisely, Table 1 reports the jump sizes for Player 1 optimal log reserves and optimal strategy corresponding to the scenario shown in Figure 3. Notice that the reserves' process jumps only when the Poisson Process  $P^1$  jumps (and since  $P^1$  has unit intensity and  $T = 2$  this happens twice on average). On the other hand, the control  $\hat{\gamma}_t^1$  being proportional to  $(\bar{X}_{t-} - X_{t-}^1)$ , it jumps every time one of the Poisson processes  $P^i$   $i = 1, \dots, N$  jumps.

Moreover, one can observe that the optimal strategies for different values of  $\lambda$  differ the most near the middle of the time interval  $[0, 2]$ . This follows from both the model's formulation and by the shape of the solution (in particular the form of the solution  $\phi$  of (26)). Indeed, in the numerical

(A) Optimal log-reserves			(B) Optimal strategies		
$\lambda=0.1$	$\lambda=0.7$	$\lambda=2$	$\lambda=0.1$	$\lambda=0.7$	$\lambda=2$
0.0238	0.0118	0.0110	0.0012	0.0019	0.0004
-0.0325	0.0728	0.0960	-0.0290	-0.0298	-0.0143
			-0.0171	-0.0176	-0.0081
			-0.0742	-0.0744	-0.0340
			-0.0072	-0.0087	-0.0043
			-0.0032	-0.0032	-0.0015
			0.0748	0.0743	0.0335
			0.0301	0.0288	0.0126
			-0.0394	-0.0393	-0.0176
			-0.1078	-0.1058	-0.0470
			0.0228	0.0226	0.0101
			-0.0664	-0.0629	-0.0275
			-0.0566	-0.0533	-0.0233
			0.0720	0.0697	0.0306
			0.0566	0.0552	0.0243
			-0.0114	-0.0192	-0.0093
			-0.1195	-0.1133	-0.0496
			-0.0395	-0.0385	-0.0171
			-0.0367	-0.0359	-0.0159
			0.0407	0.0372	0.0162
			-0.0351	-0.0347	-0.0153
			0.0471	0.0431	0.0188
			0.0697	0.0643	0.0280
			-0.0581	-0.0566	-0.0248
			-0.0326	-0.0323	-0.0142

TABLE 1. Size of the jumps. Model's parameters:  $N = 10$ ,  $T = 2$ ,  $a = 1$ ,  $\sigma = 0.8$ ,  $X_0 = 0$ ,  $\theta = 1$ ,  $\varepsilon = 10$ ,  $c = 0$ .

experiments we performed all the banks have the same initial value  $X_0$ , which then coincides with the initial mean value  $\bar{X}_0$ . Therefore the optimal strategies, which are proportional to the difference  $(\bar{X}_{t-} - X_{t-}^i)$ , are very small for  $t \sim 0$  regardless the value taken by  $\lambda$ . Furthermore,  $\phi$  has the same final value  $c$ , i.e.  $\phi(T) = c$  independently of  $\lambda$ . This makes the dependence of the optimal strategy on  $\lambda$  when  $t \sim T$  very weak. In this respect, see Figure 4 showing the behaviour of  $\phi(\lambda)$  for different values of  $\lambda$ .

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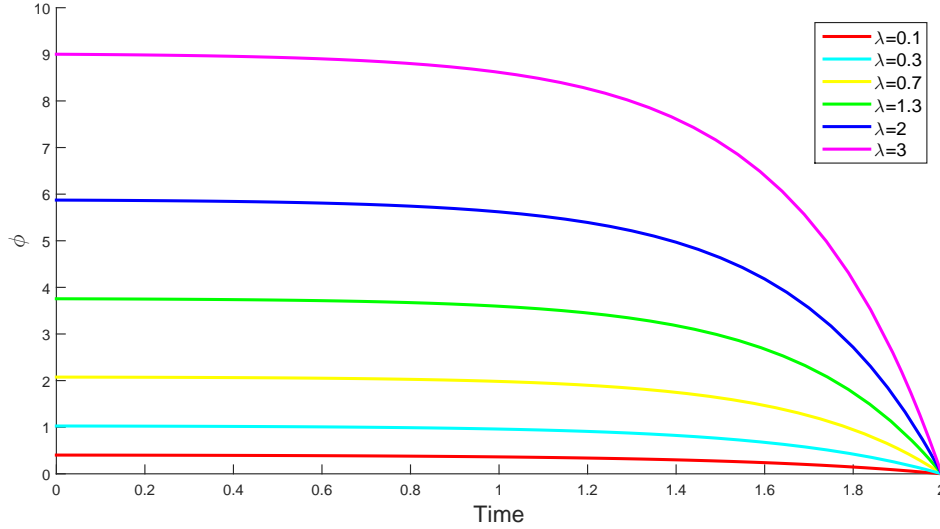


FIGURE 4. Plots of  $\phi$ , solution of the ODE (26) for different values of  $\lambda$ . Model's parameters:  $N = 10$ ,  $T = 2$ ,  $a = 1$ ,  $\theta = 1$ ,  $\varepsilon = 10$ ,  $c = 0$ .

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## APPENDIX A. INTERMEDIATE RESULTS

This appendix collects useful results which were used in the proof of Theorems 3.1 and 3.2. In the following, we will write  $|Y|_t^*$  as a shortcut for  $\sup_{s \in [0, t]} |Y_s|$ .

**Lemma A.1.** *Under Assumption A, there exists a constant  $C = C(T, c_1, c_\lambda, \chi)$  such that for any  $m \in D$  and  $P \in \mathcal{R}(m)$  we have*

$$(33) \quad \mathbb{E}^P \left[ (|X|_T^*)^2 \right] \leq C(T, c_1, c_\lambda, \chi).$$

*Proof.* Consider a given càdlàg function  $m \in D$  and a related admissible law  $P \in \mathcal{R}(m)$ . By Lemma 2.1, there exists a constant  $C$  such that

$$(34) \quad |X_t|^2 \leq C |X_0|^2 + C \left| \int_0^t b(s, X_s, m(s)) ds \right|^2 + C \left| \int_0^t \sigma(s, X_s) dB_s \right|^2 + C \left| \int_{[0, t] \times A} \lambda(m(s-)) \alpha \tilde{N}(ds, d\alpha) \right|^2.$$

In what follows, the value of the constant  $C$  may change from line to line, however we will indicate what it depends on. By Jensen’s inequality, for all  $t \in [0, T]$ ,

$$\begin{aligned} \left| \int_0^t b(s, X_s, m(s)) ds \right|^2 &\leq t \int_0^t |b(s, X_s, m(s))|^2 ds \\ &\leq t \int_0^t \sup_{0 \leq u \leq s} |b(u, X_u, m(u))|^2 ds. \end{aligned}$$

Since the function  $b$  is bounded (ref. Assumption A), we obtain

$$\left| \int_0^t b(s, X_s, m(s)) ds \right|^2 \leq c_1^2 t^2 = C(t, c_1).$$

By the Burkholder-Davis-Gundy inequality (see [Pro90, Theorem 48, Ch. IV.4]), the expected supremum of the Itô integral in (34) can be bounded as follows

$$\begin{aligned}\mathbb{E}^P \left[ \left( \left| \int_0^u \sigma(s, X_s) dB_s \right|_t^* \right)^2 \right] &\leq C \mathbb{E} \left[ \left( \int_0^t |\sigma^2(s, X_s)| ds \right) \right] \\ &\leq C t c_1^2 = C(t, c_1).\end{aligned}$$

Lastly, we have to consider the Poisson integral  $I_t = \int_{[0,t] \times A} \alpha \lambda(m(s-)) \tilde{N}(ds, d\alpha)$  in (34). Using the Burkholder-Davis-Gundy inequality, see [Pro90, Theorem 48, Ch. IV.4], it follows that

$$\mathbb{E}^P \left[ \sup_{0 \leq u \leq t} \left| \int_{[0,u] \times A} \alpha \lambda(m(s-)) \tilde{N}(ds, d\alpha) \right|^2 \right] \leq C(\theta) \mathbb{E} \left[ \int_0^t \int_A |\alpha|^2 \lambda(m(s-))^2 \Gamma_t(d\alpha) ds \right]$$

and since  $A$  is compact and  $\lambda$  is a bounded function, it is found that

$$\mathbb{E}^P \left[ (|I|_t^*)^2 \right] \leq C c_\lambda^2 \alpha_M^2 t = C(t, c_1, c_\lambda).$$

Combining all the previous estimates, we get that there exists a positive constant  $C(t, c_1, c_\lambda, \chi)$  such that

$$\mathbb{E}^P \left[ (|X|_t^*)^2 \right] \leq C(t, c_1, c_\lambda, \chi).$$

□

**Proposition A.2.** *Let  $\chi \in \mathcal{P}^2(\mathbb{R})$  and define  $\mathcal{Q} \subset \mathcal{P}(\Omega[A])$  as the set of laws  $P$  of  $\Omega[A]$ -valued random variables defined on some filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  such that:*

- (1)  $dX_t = b(t, X_t, m(t))dt + \sigma(t, X_t)dW_t + \int_A \lambda(m(t-))\alpha \tilde{N}(dt, d\alpha);$
- (2)  $X_0 \sim \chi;$
- (3)  $b: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\sigma: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  are measurable functions, bounded by some constant  $c_1;$
- (4)  $\lambda: \mathbb{R} \rightarrow \mathbb{R}_+$  is a measurable function, bounded by some constant  $c_\lambda.$

Hence  $\mathcal{Q}$  is relatively compact in  $\mathcal{P}(\Omega[A])$ .

*Proof.* Prokhorov's theorem (cf. [Bil13, Theorem 5.1, Theorem 5.2]), ensures that since  $(D, d^\circ)$  is a separable complete metric space, a family of probability measures on  $D$  is relatively compact if and only if it is tight. In order to prove the tightness, we will use the Aldous's criterion provided in [Bil13, Theorem 16.10]. By previous Lemma A.1, we have that

$$\mathbb{E}^P \left[ (|X|_T^*)^2 \right] \leq C = C(T, c_1, c_\lambda, \chi),$$

which means that  $\mathbb{E}^P \left[ (|X|_T^*)^2 \right]$  is bounded by a constant which depends upon  $P$  only through the initial distribution  $\chi$ , which is the same for all  $P \in \mathcal{Q}$ . Therefore

$$(35) \quad \sup_{P \in \mathcal{Q}} \mathbb{E}^P [(|X|_T^*)^2] \leq C < \infty.$$

Then we are left with proving that

$$(36) \quad \lim_{\delta \downarrow 0} \sup_{P \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}_T} \mathbb{E}^P \left[ |X_{(\tau+\delta) \wedge T} - X_\tau|^2 \right] = 0,$$

where  $\mathcal{T}_T$  denotes the family of all stopping times with values in  $[0, T]$  a.s. For each  $P \in \mathcal{Q}$  and each stopping time  $\tau \in \mathcal{T}_T$ , there exists a constant  $\tilde{C}$  such that

$$(37) \quad \mathbb{E}^P \left[ |X_{(\tau+\delta) \wedge T} - X_\tau|^2 \right] \leq \tilde{C} \mathbb{E}^P \left[ \left| \int_\tau^{(\tau+\delta) \wedge T} b(t, X_t, m(t)) dt \right|^2 \right] + \tilde{C} \mathbb{E} \left[ \left| \int_\tau^{(\tau+\delta) \wedge T} \sigma(t, X_t) dW_t \right|^2 \right] \\ + \tilde{C} \mathbb{E}^P \left[ \left| \int_{[\tau, (\tau+\delta) \wedge T] \times A} \lambda(m(t-)) \alpha \tilde{N}(dt, d\alpha) \right|^2 \right].$$

Without loss of generality, we may assume that  $\delta \leq 1$ . By applying Burkholder-Davis-Gundy inequality as in the proof of Theorem A.1, there exists a constant  $C$  such that

$$(38) \quad \mathbb{E}^P \left[ |X_{(\tau+\delta) \wedge T} - X_\tau|^2 \right] \leq \tilde{C} (c_1^2 \delta^2 + c_1^2 \delta + C c_\lambda^2 \alpha_M^2 \delta) \\ \leq \bar{C} (c_1, c_\lambda, \sigma) \delta.$$

Since  $\bar{C}$  does not depend on the particular choice of  $P \in \mathcal{Q}$  and of the stopping time  $\tau \in \mathcal{T}_T$  either, we have that

$$\limsup_{\delta \downarrow 0} \sup_{P \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}_T} \mathbb{E}^P \left[ |X_{(\tau+\delta) \wedge T} - X_\tau|^2 \right] \leq \limsup_{\delta \downarrow 0} \sup_{P \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}_T} \bar{C} \delta = 0.$$

Hence Aldous' criterion applies and the proof is completed.  $\square$

**Lemma A.3.** *Let  $\phi: [0, T] \times \mathbb{R}^2 \times A \rightarrow \mathbb{R}$  be a measurable, bounded function with  $\phi(t, \cdot)$  continuous for each  $t \in [0, T]$ . Then*

$$(39) \quad \Phi: D \times \mathcal{V} \times D \rightarrow \mathbb{R} \\ (m, \Gamma, X) \rightarrow \int_{[0, T] \times A} \phi(t, X_t, m(t), \alpha) \Gamma_t(d\alpha) dt$$

is a continuous function.

*Proof.* Note that the boundedness assumption, i.e.  $\|\phi\|_\infty \leq C_\phi$  for a positive constant  $C_\phi$ , guarantees that the map  $\Phi$  is well-defined since the integral domain is compact.

We need to show that when  $(\Gamma^n, X^n, m^n) \rightarrow (\Gamma, X, m)$  in  $\mathcal{P} \times D([0, T]; \mathbb{R}^2)$ , with respect to the weak convergence of measures and the Skorohod metric resp. (see [Bil13, pp. 124]), then

$$\left| \int_{[0, T] \times A} \phi(t, X_t^n, m^n(t), \alpha) \Gamma^n(dt, d\alpha) - \int_{[0, T] \times A} \phi(t, X_t, m(t), \alpha) \Gamma(dt, d\alpha) \right| \rightarrow 0.$$

Note that

$$\int_{[0, T] \times A} \phi(t, X_t, m(t), \alpha) \Gamma(dt, d\alpha) = \int_{[0, T] \times \mathbb{R}^2 \times A} \phi(t, e, \alpha) \delta_{(X_t, m(t))}(de) \Gamma(dt, d\alpha).$$



Therefore we have

$$\begin{aligned}
& \left| \int_{[0,T] \times A} \phi(t, X_t^n, m^n(t), \alpha) \Gamma^n(dt, d\alpha) - \int_{[0,T] \times A} \phi(t, X_t, m(t), \alpha) \Gamma(dt, d\alpha) \right| \\
&= \left| \int_{[0,T] \times \mathbb{R}^2 \times A} \phi(t, e, \alpha) \delta_{(X_t^n, m^n(t))}(de) \Gamma^n(dt, d\alpha) - \int_{[0,T] \times \mathbb{R}^2 \times A} \phi(t, e, \alpha) \delta_{(X_t, m(t))}(de) \Gamma(dt, d\alpha) \right| \\
&\leq \int_{[0,T] \times \mathbb{R}^2 \times A} \|\phi\|_\infty |\delta_{(X_t^n, m^n(t))}(de) \Gamma_t^n(d\alpha) - \delta_{(X_t, m(t))}(de) \Gamma_t(d\alpha)| dt \\
&\leq \int_{[0,T] \times \mathbb{R}^2 \times A} C_\phi |\delta_{(X_t^n, m^n(t))}(de) - \delta_{(X_t, m(t))}(de)| \Gamma_t^n(d\alpha) dt \\
&\quad + \int_{[0,T] \times \mathbb{R}^2 \times A} C_\phi |\Gamma_t^n(d\alpha) - \Gamma_t(d\alpha)| \delta_{(X_t, m(t))}(de) dt.
\end{aligned}$$

Since Wasserstein convergence implies weak convergence (see [Vil08, Theorem 6.9])

$$\int_{[0,T] \times \mathbb{R}^2 \times A} C_\phi \delta_{(X_t, m(t))}(de) |\Gamma_t^n(d\alpha) - \Gamma_t(d\alpha)| \leq \int_{[0,T] \times A} C_\phi |\Gamma_t^n(d\alpha) - \Gamma_t(d\alpha)| \rightarrow 0.$$

Furthermore the convergence  $(X^n, m^n) \rightarrow (X, m)$  for the Skorohod topology in  $D([0, T]; \mathbb{R}^2)$  implies convergence almost everywhere, hence

$$\delta_{(X_t^n, m^n(t))}(de) \rightarrow \delta_{(X_t, m(t))}(de), \quad \text{a.e. } t \in [0, T],$$

and therefore

$$\begin{aligned}
& \int_{[0,T] \times \mathbb{R}^2 \times A} C_\phi |\delta_{(X_t^n, m^n(t))}(de) - \delta_{(X_t, m(t))}(de)| \Gamma_t^n(d\alpha) dt \\
&= \int_{[0,T] \times \mathbb{R}^2} C_\phi |\delta_{(X_t^n, m^n(t))}(de) - \delta_{(X_t, m(t))}(de)| dt \rightarrow 0.
\end{aligned}$$

□

**Lemma A.4.** *Under Assumption A, the set-valued correspondence  $\mathcal{R}$  given in Definition 1 is continuous with relatively compact image  $\mathcal{R}(D)$  in  $\mathcal{P}(\Omega[A])$ .*

*Proof.* Using [Lac15, Lemma A.2], we prove that  $\mathcal{R}(D)$  is relatively compact in  $\mathcal{P}(\Omega[A])$  by proving that  $\{P \circ \Gamma^{-1} : P \in \mathcal{P}(\Omega[A])\}$  and  $\{P \circ X^{-1} : P \in \mathcal{P}(\Omega[A])\}$  are relatively compact sets in  $\mathcal{P}(\mathcal{V})$  and  $\mathcal{P}(D)$  respectively. The compactness of  $\{P \circ \Gamma^{-1} : P \in \mathcal{P}(\Omega[A])\}$  in  $\mathcal{P}(\mathcal{V})$  equipped with the weak convergence topology follows from the compactness of  $A$ , and therefore of  $\mathcal{V}$ . On the other hand, the compactness of  $\{P \circ X^{-1} : P \in \mathcal{P}(\Omega[A])\}$  follows from Proposition A.2.

In order to show that  $\mathcal{R}$  is upper hemicontinuous, we use the closed graph theorem (see, e.g., [AB06, Theorem 17.11]) by proving that  $\mathcal{R}$  is closed, i.e. for each  $m^n \rightarrow m \in D$  and for each convergent sequence  $P^n \rightarrow P$  with  $P^n \in \mathcal{R}(m^n)$ , it holds that  $P \in \mathcal{R}(m)$ . According to Definition 1, we have to show that  $P \circ X_0^{-1} = \chi$  and that  $\mathcal{M}^{m, \phi}$  defined in (6) is a  $P$ -martingale for all  $\phi \in C_0^\infty$ .

The first condition is satisfied since convergence in probability implies convergence in distribution and therefore  $X_0 \stackrel{d}{=} \lim_{n \rightarrow \infty} X_0^n$ , whose law is given by  $\chi$ .

Regarding the second condition, let  $s, t \in [0, T]$  be such that  $0 \leq s \leq t \leq T$ , and let  $h$  be a continuous, bounded function. By the martingale property of  $\mathcal{M}^{m^n, \phi}$ , we have  $\mathbb{E}^P[(\mathcal{M}_t^{m^n, \phi} -$

$\mathcal{M}_s^{m^n, \phi} h] = 0$  for all  $n$ . Hence to prove that  $\mathcal{M}^{m, \phi}$  is a  $P$ -martingale for all  $\phi \in C_0^\infty$  it suffices to prove

$$(40) \quad \lim_{n \rightarrow \infty} \mathbb{E}^P[(\mathcal{M}_t^{m_n, \phi} - \mathcal{M}_s^{m_n, \phi})h] = \mathbb{E}^P[(\mathcal{M}_t^{m, \phi} - \mathcal{M}_s^{m, \phi})h].$$

First of all, note that the functional  $\mathcal{M}^{m, \phi}$  can be bounded, uniformly on  $n$ . Indeed, by Taylor's theorem, we have

$$\begin{aligned} |L\phi(t, x, m, M)| &\leq \max_{x \in \mathbb{R}} \left\{ |\phi'(x)| |b(t, x, m)| + \frac{1}{2} \sigma^2(t, x) |\phi''(x)| + \int_A |\phi(x + \alpha) - \phi(x) - \phi'(x)\alpha| \Gamma_t(d\alpha) \right\} \\ &\leq \max_{x \in \mathbb{R}} \{ |\phi'(x)| |b(t, x, m)| \} + \max_{x \in \mathbb{R}} \left\{ \left| \frac{1}{2} \sigma^2(t, x) \phi''(x) \right| \right\} + \int_{A_1} \max_{x \in \mathbb{R}} \frac{|\phi''(x)|^2}{2} \alpha^2 \Gamma_t(d\alpha) \\ &\leq C(\phi', \phi'') \left( \max_{x \in \mathbb{R}} |b(t, x, m)| + \max_{x \in \mathbb{R}} \sigma^2(t, x) + \int_A \alpha^2 \Gamma_t(d\alpha) \right) \\ &\leq C(c_1, \alpha_M, \phi', \phi'') = C_\phi, \end{aligned}$$

and therefore

$$\|\mathcal{M}^{m_n, \phi}\| \leq \max_{x \in \mathbb{R}} |\phi(x)| + TC_\phi = \bar{C}_\phi.$$

Applying Lemma A.3 we have

$$\lim_{n \rightarrow \infty} \mathbb{E}^{P^n} \left[ \left( \int L\phi(s, X_s^n, m^n(s)) \right) h \right] = \mathbb{E}^P \left[ \left( \int L\phi(s, X_s, m(s)) \right) h \right]$$

and moreover, since  $P^n \rightarrow P$  and  $\phi$  is bounded and continuous, we have that

$$\mathbb{E}^{P^n}[\phi(X_t^n)] = \int \phi(X_t) dP^n \rightarrow \int \phi(X_t) dP = \mathbb{E}^P[\phi(X_t)].$$

Therefore we can conclude that for each continuous, bounded function  $h$

$$\mathbb{E}^P[(\mathcal{M}_t^{m, \phi} - \mathcal{M}_s^{m, \phi})h] = \lim_{n \rightarrow \infty} \mathbb{E}^P[(\mathcal{M}_t^{m_n, \phi} - \mathcal{M}_s^{m_n, \phi})h] = 0,$$

which implies  $\mathbb{E}^P[\mathcal{M}_t^{m, \phi} - \mathcal{M}_s^{m, \phi}] = 0$ , i.e.  $\mathcal{M}^{m, \phi}$  is a martingale and  $P \in \mathcal{R}(m)$ .

We are left with proving that  $\mathcal{R}$  is also lower hemicontinuous. Let  $m \in D$  and  $m^n$  be a sequence of càdlàg functions converging to  $m$ . Then, for every  $P \in \mathcal{R}(m)$  we need to exhibit a sequence  $P_n \in \mathcal{R}(m^n)$  such that  $P_n \rightarrow P$  in  $\mathcal{P}(\Omega[A])$ . Let  $(\Omega', \mathcal{F}', \mathbb{F}', P')$  be a filtered probability space,  $W$  a  $\mathbb{F}'$ -Brownian motion and  $N$  a Poisson random measure on  $[0, T] \times A$  with intensity measure  $\mathcal{L} \times \Gamma_t$ . Let  $X$  be the unique strong solution of the following SDE:

$$(41) \quad dX_t = b(t, X_t, m(t))dt + \sigma(t, X_t)dW_t + \int_A \alpha \lambda(m(t-))N(dt, d\alpha)$$

with initial condition  $X_0$ . The existence and uniqueness of strong solution of equation (41) is guaranteed by [App09, Theorem 6.2.3]. Then by uniqueness we have  $P' \circ (\Gamma, X)^{-1} = P$ . For each  $n$ , define  $X^n$  as the process solving

$$dX_t^n = b(t, X_t^n, m^n(t))dt + \sigma(t, X_t^n)dW_t + \int_A \alpha \lambda(m^n(t-))N(dt, d\alpha), \quad X_0^n = X_0.$$

We want to show that

$$(42) \quad \mathbb{E}^{P'} \left[ (|X^n - X_t^*|)^2 \right] \rightarrow 0.$$

For a suitable positive constant  $C > 0$  we have

$$(43) \quad |X_t^n - X_t|^2 \leq C \left( \int_0^t |b(s, X_s^n, m^n(s)) - b(s, X_s, m(s))| ds \right)^2 + C \left| \int_0^t |\sigma(s, X_s^n) - \sigma(s, X_s)| dW_s \right|^2 + C \left| \int_A \alpha(\lambda(m^n(t-)) - \lambda(m(t-))) \tilde{N}(dt, d\alpha) \right|^2.$$

Since  $b$  is Lipschitz continuous in  $x$ , we have that

$$(44) \quad \mathbb{E}^{P'} \left[ \int_0^t |b(s, X_s^n, m^n(s)) - b(s, X_s, m(s))|^2 ds \right] \leq c_1 \int_0^t \mathbb{E}^{P'} (|X^n - X|_s^*)^2 ds + \mathbb{E}^{P'} \left[ \int_0^t |b(s, X_s, m^n(s)) - b(s, X_s, m(s))|^2 ds \right].$$

Since  $b$  is a continuous function, and  $m^n, m$  are càdlàg functions, then also  $b \circ m^n$  and  $b \circ m$  are in  $D$ . By construction,  $m^n \rightarrow m$  in the Skorohod topology, and this implies that  $m^n \rightarrow m$  a.e. for the Lebesgue measure (see [Bil13, pp. 124]) and therefore  $b \circ m^n \rightarrow b \circ m$  a.e. for the Lebesgue measure. In particular since  $b$  is bounded, by applying the dominated convergence theorem we obtain that the second expectation in the RHS of (44) vanishes as  $m^n \rightarrow m$ .

Regarding the stochastic integral in (43), we can apply Jensen and Burkholder-Davis-Gundy inequalities to obtain

$$\begin{aligned} \mathbb{E}^{P'} \left[ \left| \int_0^u |\sigma(s, X_s^n) - \sigma(s, X_s)| dW_s \right|^2 \right] &\leq C \mathbb{E}^{P'} \left[ \int_0^t c_1 |X_s^n - X_s|^2 ds \right] \\ &\leq C c_1 \int_0^t \mathbb{E}^{P'} [|X^n - X|_s^*] ds. \end{aligned}$$

Lastly, applying the Burkholder-Davis-Gundy inequality to the Poisson integral in (43), we have that

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_A \alpha(\lambda(m^n(t-)) - \lambda(m(t-))) \tilde{N}(dt, d\alpha) \right|^2 \right] &\leq \bar{C} \mathbb{E} \left[ \int_{[0, T] \times A} |\alpha|^2 |\lambda(m^n(t-)) - \lambda(m(t-))|^2 \Gamma_t(d\alpha) dt \right] \\ &\leq \bar{C} \alpha_M^2 \int_{[0, T]} |\lambda(m^n(t)) - \lambda(m(t))|^2 dt, \end{aligned}$$

where the RHS converges to zero when  $m^n \rightarrow m$ . Indeed, arguing as before, since  $\lambda$  is a continuous, bounded function (Assumption (A.1)) and  $m^n \rightarrow m$  in the Skorohod topology,  $\lambda \circ m^n \rightarrow \lambda \circ m$  a.e. and therefore the dominated convergence theorem implies that  $\int_0^t |\lambda(m^n(s)) - \lambda(m(s))|^2 ds \rightarrow 0$ , as  $n \rightarrow \infty$ .

Therefore, combining the previous results and applying the Gronwall's inequality, we conclude that  $\mathbb{E}^{P'} [|X^n - X|_T^*]^2 \rightarrow 0$ .

By defining  $P^n = P \circ (\Gamma, X^n)^{-1}$  we have found the desired sequence such that  $P^n \rightarrow P$  in  $\mathcal{P}(\Omega)$ . To conclude, we have to show that  $P^n$  is indeed in  $\mathcal{R}(m^n)$ .  $P^n$  satisfies condition (1) in Definition 1 by construction, and condition (2) can be checked by applying Itô's formula to  $\phi(X^n)$ , for each  $\phi \in C_b^\infty(\mathbb{R})$ .  $\square$

**Lemma A.5.** *The operator  $J$  defined in (10) is continuous under assumptions A.*

*Proof.* Since  $f$  and  $g$  are bounded functions, then also the functional  $\mathcal{C}^m$  (see (8)) is uniformly bounded. Then in order to prove that the functional  $J$  is continuous, it is sufficient to show that  $(D \times \Omega[A]) \ni (m, \Gamma, X) \mapsto \mathcal{C}^m(\Gamma, X)$  is continuous.

Moreover, since by Assumption A the cost functions  $f: [0, T] \times \mathbb{R} \times \mathbb{R} \times A \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous in  $(x, m, \alpha)$  and  $(x, m)$  respectively, the continuity of the function

$$(m, \Gamma, X) \mapsto \int_{[0, T] \times A} f(t, X_t, m(t), \alpha) \Gamma(dt, d\alpha) + g(X_T, m(T))$$

follows from Lemma A.3.  $\square$

## APPENDIX B. ON MARKOVIAN PROJECTIONS OF SEMIMARTINGALES WITH JUMPS

In this part of the appendix, we show how to slightly relax the continuity assumption in [BC09, Theorems 1 and 2], so that the same conclusions hold and can be safely used in the proof of our Theorem 3.2. First, we observe that the results in [BC09, Theorems 1 and 2] are proved under the assumption that the compensator  $n(t, x, d\alpha)$  satisfies the following continuity property:

**Assumption (BC).** Let  $n(t, \cdot)(I)$  be continuous in  $\mathbb{R}$  uniformly in  $t$  and let  $n(\cdot, x)(I)$  be right-continuous on  $[0, T[$  uniformly in  $x$  for each  $I \in \mathcal{B}(A)$ .

However the same results can be obtained by assuming only continuity for the weak convergence of measures. Recall that, given a measure over the product space  $W \times K$ , with  $W$  and  $K$  metric spaces,  $n(\omega, dy)$  is weak continuous if

$$\int_K f(y) n(\omega_n, dy) \rightarrow \int_K f(y) n(\omega, dy), \quad \text{for all } f \in C_b(K),$$

when  $\omega_n \rightarrow \omega$ . Let

$$w(t, x) = \int_K [\phi(x + \alpha) - f(x) - \alpha \phi'(x)] n(t, x, d\alpha).$$

Then if  $(t_n, x_n) \rightarrow (t, x) \in [0, T] \times \mathbb{R}$ , where  $t_n$  is a decreasing sequence, we have

$$\begin{aligned} |w(t_n, x_n) - w(t, x)| &= \left| \int_K [\phi(x_n + \alpha) - \phi(x_n) - \alpha \phi'(x_n)] n(t_n, x_n, d\alpha) \right. \\ &\quad \left. - \int_K [\phi(x + \alpha) - \phi(x) - \alpha \phi'(x)] n(t, x, d\alpha) \right| \\ &\leq \left| \int_K [\phi(x_n + \alpha) - \phi(x_n) - \alpha \phi'(x_n) - (\phi(x + \alpha) - f(x) - \alpha \phi'(x))] n(t_n, x_n, d\alpha) \right| \\ (45) \quad &\quad + \left| \int_K [\phi(x + \alpha) - \phi(x) - \alpha \phi'(x)] (n(t_n, x_n, d\alpha) - n(t, x, d\alpha)) \right|. \end{aligned}$$

Both integrals in the RHS of equation (45) vanish when  $n \rightarrow \infty$ : the first one via the dominated convergence theorem, since  $\phi$  and  $\phi'$  are bounded in the compact domain  $[0, T] \times A$ , the second one by definition of weak convergence. One of the main steps in the proofs of [BC09, Theorems 1 and 2] is [BC09, Proposition 1]. We show that such a proposition still holds after replacing Assumption (BC) with the following:

**Assumption (wBC).** Let  $n(t, \cdot, d\alpha)$  be weak continuous in  $\mathbb{R}$  and let  $n(\cdot, x, d\alpha)$  be right weak continuous in  $[0, T[$

In order to do so, we need to show that the martingale problem for  $(\mathcal{L}_t)_{t \in [0, T]}$  and any initial condition  $(t_0, x_0)$ , where

$$\begin{aligned} \mathcal{L}_t \phi(x) &= b(t, x, m(t)) \phi'(x) + \frac{1}{2} \sigma^2(t, x) \phi''(x) \\ &\quad + \lambda(m(t-)) \int_A [\phi(x + \alpha) - \phi(x) - \phi'(x) \alpha] n(t, x, \alpha) d\alpha \end{aligned}$$

is well-posed, and its unique solution  $Q_{t_0, x_0}$  satisfies

$$(46) \quad \lim_{t \downarrow t_0} \mathbb{E}^{Q_{t_0, x_0}}[\phi(X_t)] = \phi(x_0) \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}).$$

The well-posedness of the martingale problem follows by [MP92, Theorem 5] where no continuity assumptions on the compensator are required. Moreover  $\mathcal{L}_s \phi(X_s)$  is uniformly bounded in  $s$  by the boundedness assumption on the coefficients (ref. Assumption A) and the finiteness of the measure  $n$ , and it is right-continuous by equation (45). Therefore the continuity property (46) follows.

Furthermore [BC09, Theorem 1, Theorem 2] still hold under Assumption (wBC). The former, i.e. [BC09, Theorem 1], assures that the law of  $X$  under the martingale problem's solution  $Q_{t_0, x_0}$  satisfies the Kolmogorov equation

$$\int_{\mathbb{R}} \phi(y) q_{t_0, t}(x_0, dy) = \phi(x_0) + \int_{t_0}^t \int_{\mathbb{R}} \mathcal{L}_s \phi(y) q_{t_0, s}(x_0, dy) ds, \quad g \in C_0^\infty(\mathbb{R}),$$

with initial condition  $q_{t_0, t_0}(x_0, dy) = \delta_{x_0}(dy)$ , which is still verified since  $t \mapsto \mathcal{L}_s \phi(X_s)$  is right-continuous and uniformly bounded, hence

$$\int_{\mathbb{R}} g(y) q_{t_0, t_0}(x_0, dy) = g(x_0) = \int_{\mathbb{R}} g(y) \delta_{x_0}(dy), \quad g \in C_0^\infty(\mathbb{R}).$$

The latter, i.e. [BC09, Theorem 2], provides a Markovian projection for the state process, and its proof is based on the previous results and not directly on Assumption (BC).

#### APPENDIX C. DETAILS ON THE COMPUTATION OF $\phi$ IN EQUATION (26)

We need to solve the final value Cauchy problem

$$\begin{cases} \dot{\phi}(t) = F(\phi(t)) \\ \phi(T) = c > 0 \end{cases}$$

where

$$F(u) = \frac{Au^2 + Bu + C}{1 + ku}.$$

Then, if we perform the time reversal  $\tau = T - t$  we can consider the equivalent Cauchy problem

$$\begin{cases} \dot{\phi}(\tau) = -F(\phi(\tau)) \\ \phi(0) = c > 0. \end{cases}$$

Moreover we look for a solution whose graph belongs to the domain  $D_k := [0, T] \times (-\frac{1}{k}, \infty)$ , where  $k := \lambda(1 - \frac{1}{N})^2$ , so that the optimal strategy given in (24) is well defined for all  $t \in [0, T]$ . Notice that  $k \neq 0$  if  $\lambda > 0$  and  $N \geq 2$ , which we can safely assume to rule out trivialities.

Note that since  $-F$  and  $-\dot{F}$  are continuous functions (in the smaller domain  $D_k$ ) then we can apply standard results for existence and uniqueness. Therefore the function  $\phi$  solves the following Cauchy problem

$$\begin{cases} (1 + k\phi_t)\dot{\phi}_t = A\phi_t^2 + B\phi_t + C \\ \phi_T = c \end{cases}$$

for appropriate values of  $k, A, B, C$  which do not depend on time  $t$ :

$$\begin{aligned} A &:= \left(1 + 2a \left(1 - \frac{1}{N}\right)\right) \lambda \left(1 - \frac{1}{N}\right) > 0, \\ B &:= \lambda \theta \left(2 - \frac{1}{N}\right) - \varepsilon \lambda^2 \left(1 - \frac{1}{N}\right)^2 + 2a, \\ C &:= \lambda(\theta^2 - \varepsilon) < 0. \end{aligned}$$

Let

$$\omega(t) := \left(\phi(t) + \frac{1}{k}\right) e^{-\frac{A}{k}t}.$$

Then  $\omega$  solves the following ODE:

$$\dot{\omega}\omega = F_1(t)\omega + F_0(t),$$

where

$$\begin{aligned} F_0(t) &= \left(\frac{C}{k} - \frac{B}{k^2} + \frac{A}{k^3}\right) e^{-2\frac{A}{k}t}, \\ F_1(t) &= \left(\frac{B}{k} - 2\frac{A}{k^2}\right) e^{-\frac{A}{k}t}. \end{aligned}$$

Lastly, by choosing

$$\xi = \int F_1(t) dt = \left(\frac{2}{k} - \frac{B}{A}\right) e^{-\frac{A}{k}t},$$

we have that

$$(47) \quad \omega(\xi)\dot{\omega}(\xi) = \omega(\xi) + K\xi,$$

where

$$K = -\frac{(Ck^2 - Bk + A)A}{(2A - kB)^2}.$$

Equation (47) can be solved in parametric form as

$$\begin{cases} \xi = h \exp\left(-\int \frac{\tau}{\tau^2 - \tau - K} d\tau\right), \\ \omega = h\tau \exp\left(-\int \frac{\tau}{\tau^2 - \tau - K} d\tau\right), \end{cases}$$

where  $h$  is a suitable constant. Therefore,

$$\begin{cases} \xi = h \exp\left(-\int \frac{\tau}{\tau^2 - \tau - K} d\tau\right), \\ \phi = \left(h\tau \exp\left(-\int \frac{\tau}{\tau^2 - \tau - K} d\tau\right)\right) e^{\frac{A}{k}t - \frac{1}{k}}. \end{cases}$$

Then, computing the integral in the definition of  $\xi$  gives

$$\xi = h \exp \frac{\left( -\frac{\tan^{-1} \left[ \frac{-1+2\tau}{\sqrt{-1-4K}} \right]}{\sqrt{-1-4K}} \right)}{\sqrt{\tau^2 - \tau - K}}.$$

Finally, in order to compute  $\phi$  explicitly, we need to invert this relation  $\xi \mapsto \xi(\tau)$  and plug it into  $\omega$ .